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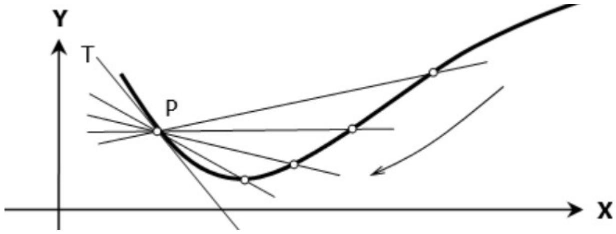
2.1 - Tangent and Velocity Problems

Review



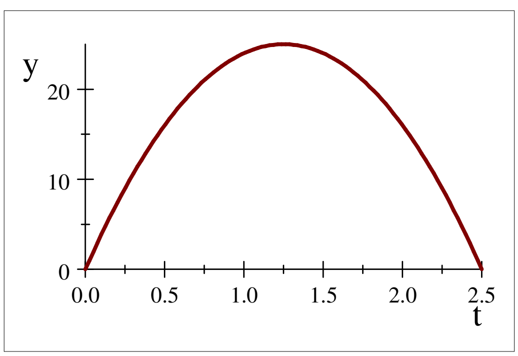
Average Tangent slope over some distance $=$ Secant Slope $= \frac{\text{Change in } y}{\text{Change in } x}$.

Tangent to a Curve: On a curve $y = f(x)$, the **slope** T **of the tangent line** at point P on the curve, is the limit of the slopes of the secant lines PQ as $Q \to P$ (see above). For example, if we label the Q points as $Q_1, Q_2, \ldots$, and let's further assume that we measure the slopes to be $T_1 = slope(PQ_1) = \frac{1}{2}$, $T_2 = slope(PQ_2) = 0$, and so on as Q_i approaches P. We might end up with the sequence of slopes: $\frac{1}{2}, 0, -1, -1.5, -1.6, -1.65, -1.68, -1.69, -1.695, -1.698, -1.6995, \ldots$ At some point, we may wish to conclude that the slopes are "approaching" -1.7. In other words, the limit of the slopes of the secant lines PQ_i is $T = slope(PQ_\infty) = -1.7$.

Velocity: Given a graph of the position function $y(t)$ of an object, the velocity $v(t)$ at time t_0 is the slope T of the tangent line at $P = \big(t_0,\ y(t_0)\big)$ (the limit of the slopes of the secant lines).

Average velocity over some time $=$ Secant Slope $= \frac{\text{Change in } y}{\text{Change in } t}$.

Problem 5. If a ball is thrown into the air with a velocity of 40 *ft*/*s*, its height in feet t seconds later is given by $y = 40t - 16t^2$.



a. Find the average velocity for the time period beginning when $t = 2$ and lasting:

0.5 seconds, 0.1 seconds, 0.05 seconds, 0.01 seconds.

At $t = 2$,
$y = 40(2) - 16(2)^2 = 16$.

Let h represent the time duration: $0.5, 0.1, 0.05$, or 0.01.

The average velocity between times 2 and $2 + h$ is …

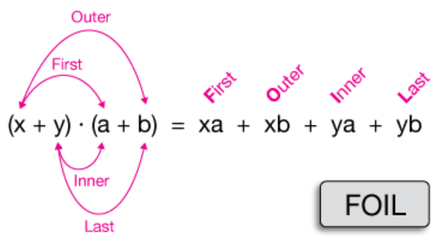
$$v_{avg} = \frac{\text{Change in } y}{\text{Change in } t} = \frac{y(2+h)-y(2)}{(2+h)-2}.$$

Since $y = 40t - 16t^2$, ...

$$v_{avg} = \frac{\left[40(2+h)-16(2+h)^2\right]-16}{h}, \text{ if } h \neq 0.$$

$$= \frac{\left[(80+40h)-16\left(4+4h+h^2\right)\right]-16}{h} = \frac{\left[(80+40h)-\left(64+64h+16h^2\right)\right]-16}{h}$$

Recall F.O.I.L (First, Outer, Inner, Last):



$$= \frac{80+40h-64-64h-16h^2-16}{h} = \frac{-24h-16h^2}{h}$$

$$= -24 - 16h.$$

Lasting: 0.5 seconds, $h = 0.5,\ v_{avg} = -24 - 16(0.5) = -32\ \frac{ft}{s}.$

Lasting: 0.1 seconds, $h = 0.1,\ v_{avg} = -24 - 16(0.1) = -\ 25.6\ \frac{ft}{s}.$

Lasting: 0.05 seconds, $h = 0.05,\ v_{avg} = -24 - 16(0.05) = -24.8\ \frac{ft}{s}.$

Lasting: 0.01 seconds, $h = 0.01,\ v_{avg} = -24 - 16(0.01) = -24.16\ \frac{ft}{s}.$

b. Estimate the instantaneous velocity when $t = 2$.

$v(2) \approx -24\ \frac{ft}{s}.$

Problem 9. The point $P(1,0)$ lies on the curve $y = \sin(\frac{10\pi}{x})$.

If Q_x is the point $(x,\ \sin(\frac{10\pi}{x}))$, find the slope of the secant line PQ_x for...
$x = 2,\ 1.5,\ 1.4,\ 1.3,\ 1.2,\ 1.1,\ 0.5,\ 0.6,\ 0.7,\ 0.8,$ and 0.9.

$$mPQ_x = \frac{y(1+\Delta x)-y(1)}{(1+\Delta x)-1}$$

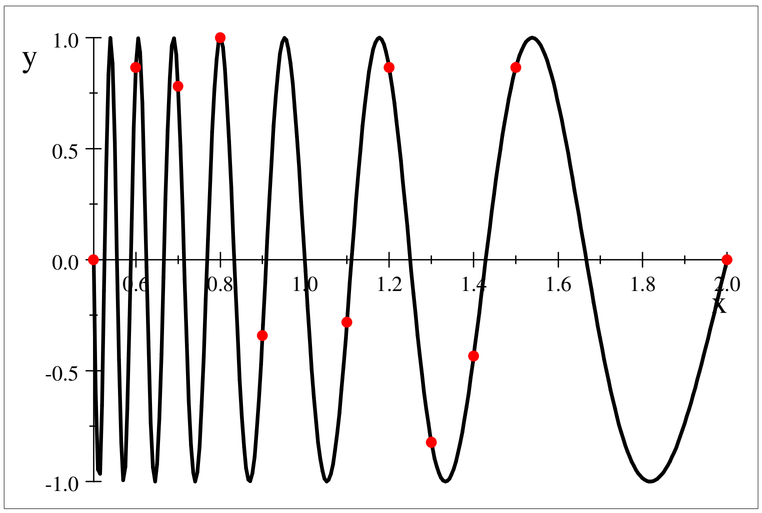
$$= \frac{\sin\left(\frac{10\pi}{1+\Delta x}\right)-0}{\Delta x} = \frac{\sin\left(\frac{10\pi}{1+\Delta x}\right)}{\Delta x}.$$

Approaching from the right		
x	Q_x	mPQ_x
2	$(2, 0)$	0
1.5	$(1.5, 0.8660)$	1.7321
1.4	$(1.4, -0.4339)$	-1.0847
1.3	$(1.3, -0.8230)$	-2.7433
1.2	$(1.2, 0.8660)$	4.3301
1.1	$(1.1, -0.2817)$	-2.8173

Approaching from the left		
x	Q_x	mPQ_x
0.5	$(0.5, 0)$	0
0.6	$(0.6, 0.8660)$	-2.1651
0.7	$(0.7, 0.7818)$	-2.6061
0.8	$(0.8, 1)$	-5
0.9	$(0.9, -0.3420)$	3.4202

Approaching a limit?

b. Use a graph of the curve to explain why the slopes of the secant lines in part *a* are not close to the slope of the tangent line at *P*.



c. By choosing appropriate secant lines, estimate the slope of the tangent line at *P*.

If we choose $x = 1.001$, then

the point Q is $(1.001, \ -0.0314)$

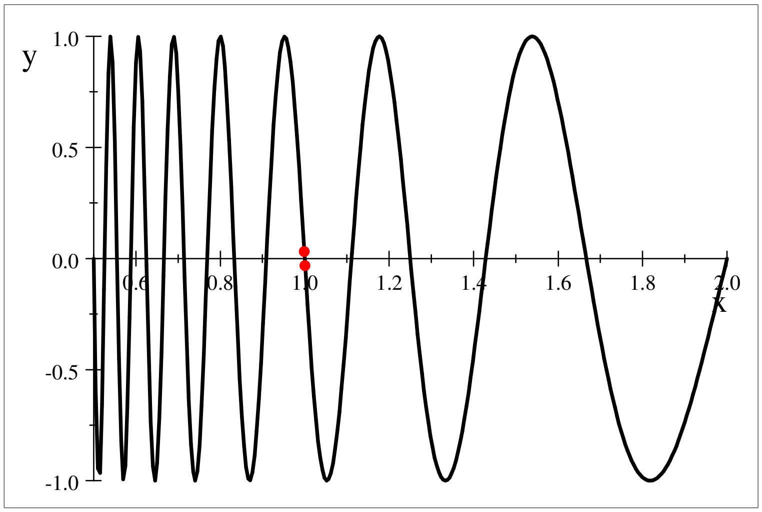
and $mPQ \approx -31.3794$.

If $x = 0.999$, then

Q is $(0.999, 0.0314)$

and $mPQ = -31.4422$.

The average of these slopes is $P \approx \frac{-31.3794-31.4422}{2} = -31.4108$.



y
1.0
0.5
0.0
-0.5
-1.0
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
x