MATH 1272: **Calculus II**

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11.5 - Alternating Series

Review:

Alternating Series: A series whose terms are alternatively positive and negative. For example: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

Alternating Series Test: Given an alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - \ldots,$ where b_n satisfies the two conditions (i) $b_{n+1} \leq b_n$ for all n, and (ii) $\lim_{n\to\infty} b_n = 0$; then the series is convergent.

Estimating Sums

Alternating Series Estimation Theorem: If $s = \Sigma(-1)^{n-1}b_n$ is the sum of an alternating series that satisfies the conditions (i) and (ii) from the alternating series test, then $|R_n| = |s - s_n| \leq b_{n+1}$.

Problem #4 Test the series $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \ldots$ for convergence or divergence.

$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n+1}}$.

Now, $b_n = \frac{1}{\sqrt{n+1}} > 0,$ $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} b_n = 0,$ so the series converges by the alternating series test.

Problem #20 Test the series $\sum_{n=1}^{\infty}(-1)^n\left(\sqrt{n+1} - \sqrt{n}\right)$ for convergence or divergence.

$b_n = \frac{\sqrt{n+1}-\sqrt{n}}{1} \bullet \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} > 0,$ for $n \geq 1$.

$\{b_n\}$ is decreasing and $\lim_{n\to\infty} b_n = 0$, so the series $\sum_{n=1}^{\infty}(-1)^n\left(\sqrt{n+1} - \sqrt{n}\right)$ converges by the alternating series test.

Problem #26 Show that the series $\sum_{n=1}^{\infty}(-1)^{n-1}ne^{-n}$ is convergent. How many terms of the series do we need to add in order to find the sum to the accuracy $|error| < 0.01$?

The series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{e^n}$ satisfies (i) of the alternating series test because

$\left(\frac{x}{e^x}\right)' = \frac{e^x(1)-xe^x}{(e^x)^2} = \frac{1-x}{e^x} < 0,$ for $x > 1$ and

(ii) $\lim_{n\to\infty}\frac{n}{e^n} = \lim_{x\to\infty}\frac{x}{e^x} \overset{L'H}{=} \lim_{x\to\infty}\frac{1}{e^x} = 0,$ so the series is convergent.

Now, $b_6 = \frac{6}{e^6} \approx 0.015 > 0.01$ and $b_7 = \frac{7}{e^7} \approx 0.006 < 0.01$.

So by the alternating series estimation theorem, $n = 6$ (That is, since the seventh term is less than the desired error, we need to add the first six terms to get the sum to the desired accuracy.).

Problem #34 For what values of p is the series $\sum_{n=2}^{\infty}(-1)^{n-1}\frac{(\ln n)^p}{n}$ convergent?

Let $f(x) := \frac{(\ln x)^p}{x}$. When does, $f'(x) = \frac{(\ln x)^{p-1}(p-\ln x)}{x^2} < 0$.

When $p < \ln x$, or $e^p < x$.

So f is eventually decreasing for every p.

Clearly, $\lim_{n\to\infty} \frac{(\ln n)^p}{n} = 0$ if $p \leq 0$.

And if $p > 0$, we can apply L'Hospital's rule:

$$\lim_{n\to\infty} \frac{(\ln n)^p}{n} = \lim_{x\to\infty} \frac{(\ln x)^p}{x} \overset{L'H}{=} \lim_{x\to\infty} \frac{p(\ln x)^{p-1}\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{p(\ln x)^{p-1}}{x} \overset{L'H}{=} \lim_{x\to\infty} \frac{p(p-1)(\ln x)^{p-2}}{x}$$

$$\overset{L'H}{=} \ldots \overset{L'H}{=} \lim_{x\to\infty} \frac{p(p-1)!}{x} = 0$$

So the series converges for all p (by the alternating series test).