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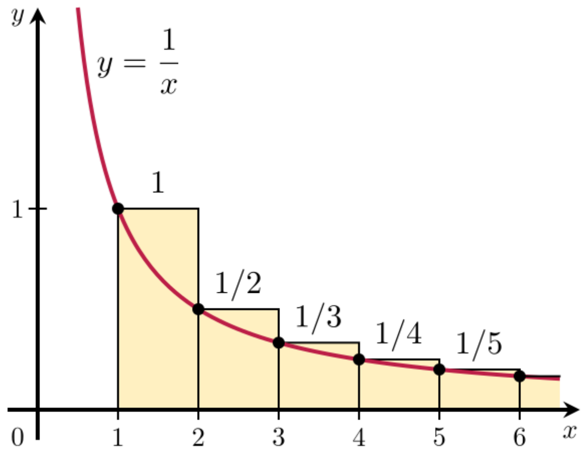
11.3 - Integral Test and Estimates of Sums

Review:

The Integral Test: Suppose f is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series Σa_n is convergent if and only if the improper integral $\int_1^\infty f(x)dx$ is convergent. In other words:

- If $\int_1^\infty f(x)dx$ is convergent, then Σa_n is convergent.
- If $\int_1^\infty f(x)dx$ is divergent, then Σa_n is divergent.



Observe that it is not necessary to start the series or the integral at $n = 1$. Also, it is not necessary that f be always decreasing, merely that it is eventually decreasing for all $x > M$, for some $M \in \mathbb{R}$.

The p-series: $\sum_{n=1}^\infty \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Caution, in general $\sum_{n=1}^\infty a_n \neq \int_1^\infty f(x)dx$.

Estimating the Sum of a Series

Remainder: $R_n := s - s_n = a_{n+1} + a_{n+2} + \ldots$
The remainder is the error made when s_n (the sum of the first n terms) is used as an approximation to the total sum.
Remainder Estimate for the Integral Test: Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and Σa_n is convergent. If $R_n = s - s_n$, then
$\int_{n+1}^\infty f(x)dx \leq R_n \leq \int_n^\infty f(x)dx.$ [Equation 3]
Adding s_n to the inequality above, we get a lower and upper bound for our sum s:
$s_n + \int_{n+1}^\infty f(x)dx \leq s \leq s_n + \int_n^\infty f(x)dx.$
This provides a more accurate approximation to the some of the series than the partial sum s_n does.

Problem #4 Use the Integral Test to determine whether the series $\sum_{n=1}^\infty \frac{1}{n^5}$ is convergent or divergent.

The function $f(x) = \frac{1}{x^5}$ is continuous, positive, and decreasing on $[1,\infty)$, so the integral test applies.

$$\int_1^\infty \frac{1}{x^5}dx = \lim_{t\to\infty} \int_1^t x^{-5}dx = \lim_{t\to\infty} \left[\frac{x^{-4}}{-4}\right]_1^t = \lim_{t\to\infty} \left(-\frac{1}{4t^4} + \frac{1}{4}\right) = \frac{1}{4}.$$

Since this improper integral is convergent, the series $\sum_{n=1}^\infty \frac{1}{n^5}$ is also convergent by the integral test.

Problem #30 Find the values of p for which the series $\sum_{n=3}^{\infty} \frac{1}{n \ln n [\ln(\ln n)]^p}$ is convergent.

$f(x) := \frac{1}{x \ln x [\ln(\ln x)]^p}$ is positive and continuous on $[3, \infty)$.

For $p \geq 0$, f clearly decreases on $[3, \infty)$; and for $p < 0$ it can be verified that f is ultimately decreasing.

Thus, we can apply the integral test.

$$I = \int_3^{\infty} \frac{dx}{x \ln x [\ln(\ln x)]^p} = \lim_{t \to \infty} \int_3^t \frac{[\ln(\ln x)]^{-p}}{x \ln x} dx = \lim_{t \to \infty} \left[\frac{[\ln(\ln x)]^{-p+1}}{-p+1} \right]_3^t \qquad (\text{for } p \neq 1)$$

$$= \lim_{t \to \infty} \left[\frac{[\ln(\ln t)]^{-p+1}}{-p+1} - \frac{[\ln(\ln 3)]^{-p+1}}{-p+1} \right],$$ which exists whenever $-p + 1 < 0$ or $p > 1$.

If $p = 1$, then $I = \lim_{t \to \infty} [\ln(\ln(\ln x))]_3^t = \infty$.

Therefore, $\sum_{n=3}^{\infty} \frac{1}{n \ln n [\ln(\ln n)]^p}$ converges for $p > 1$.

Problem #38 Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^5}$ correct to three decimal places.

$f(x) := \frac{1}{x^5}$ is positive and continuous and $f'(x) = -\frac{5}{x^6}$ is negative for $x > 0$, and so the integral test applies.

Using Equation 3, we have $R_n \leq \int_n^{\infty} x^{-5} dx = \lim_{t \to \infty} \left[-\frac{1}{4x^4} \right]_n^t = \frac{1}{4n^4}$.

If we take $n = 5$, then $s_5 = 1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \frac{1}{5^5} \approx 1.036662$ and $R_5 \leq 0.0004$.

So, $s \approx s_5 \approx 1.037$.