Discussion Instructor: Jodin Morey moreyjc@umn.edu
Discussion Session Website: math.umn.edu/~moreyjc

11.2 - Series

Review:

Series: If we try to add the terms of an infinite sequence $\{a_n\}$, we can write an expression of the form "$a_1 + a_2 + \ldots + a_n + \ldots$" which is called an **infinite series**, or just a **series**. We notate this as $\sum_{n=1}^{\infty} a_n$ or Σa_n. Observe that every real number can be expressed as an infinite series.

$0 = 0 + 0 + \ldots,$ $\quad 5 = 5 + 0 + 0 + \ldots,$

$\frac{1}{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots,$ $\quad \pi = 3 + \frac{1}{10} + \frac{4}{100} + \ldots$

Convergence of Series: Given the series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots$, let s_n denote its nth **partial sum** defined as: $s_n := \sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n$. Observe that s_n is a real number for each n. If the new sequence $\{s_n\}$ is convergent, and its limit $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series Σa_n is called **convergent**, and we write $a_1 + a_2 + \ldots + a_n + \ldots := s$ or $\sum_{n=1}^{\infty} a_n := s$. The number s is called the **sum** of the series. If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

As a mnemonic device, and to gain better understanding, observe that $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$ is similar to how $\int_1^{\infty} f(x)dx = \lim_{t\to\infty} \int_1^t f(x)dx$.

Geometric Series:
Assume $a \neq 0$, then $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$ is called a **geometric series**. r is called its **common ratio**. It's partial sum is $s_n = \frac{a(1-r^n)}{1-r}$. If $|r| < 1$, then the geometric series is convergent, and $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$. If $|r| \geq 1$, then the series is divergent. (see the text for the simple derivations of these sums, which can serve as mnemonic devices).

Telescopic Sum: A series Σa_n whose terms, when written out $(a_0 + a_1 + a_2 + \ldots)$, cancel each other leaving a finite number of terms.
Example: $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \Sigma\left(\frac{1}{i} - \frac{1}{i+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$
$= 1 - \frac{1}{n+1}.$
(notice how the second term in each parentheses cancels with the first term in the subsequent set of parentheses)

Harmonic Series: $\Sigma \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ is divergent! (see text for simple proof)

Convergent Series, **Zero Limit Theorem**: If Σa_n is convergent, then $\lim_{n\to\infty} a_n = 0$. However, it is NOT true that $\lim_{n\to\infty} a_n = 0$ implies that Σa_n is convergent. The harmonic series above is the most obvious example of this.

To avoid confusion, observe that for any series Σa_n, we often refer to two different sequences: the sequence of its terms $\{a_n\}$, and the sequence of its partial sums $\{s_n\}$. If Σa_n converges, then $\lim_{n\to\infty} s_n = s = \Sigma a_n$, and we can conclude that $\lim_{n\to\infty} a_n = 0$.

Test for Divergence: If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series Σa_n is divergent.

Linearity of Convergent Series: If Σa_n and Σb_n are convergent series, then so are $\Sigma(a_n + b_n)$, $\Sigma(a_n - b_n)$, and $\Sigma c a_n$ (where c is a constant). We also have the following:

♦ $\Sigma c a_n = c \Sigma a_n$ $\quad$ ♦ $\Sigma(a_n \pm b_n) = \Sigma a_n \pm \Sigma b_n$.

Problem #2 Explain what it means to say that $\sum_{n=1}^{\infty} a_n = 5$.

It means that by adding sufficiently many terms of the series we can get as close to 5 as we like.

Problem #4 Calculate the sum of the series $\sum_{n=1}^{\infty} a_n$ with partial sums $s_n = \frac{n^2-1}{4n^2+1}$.

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n^2-1}{4n^2+1}$$

$$= \lim_{n\to\infty} \frac{\frac{n^2-1}{n^2}}{\frac{4n^2+1}{n^2}} = \lim_{n\to\infty} \frac{1-\frac{1}{n^2}}{4+\frac{1}{n^2}}$$

$$= \frac{1-0}{4+0} = \frac{1}{4}.$$

Problem #7 For the series $\sum_{n=1}^{\infty} \frac{n}{1+\sqrt{n}}$, calculate the first eight terms of the sequence of partial sums correct to four decimal places. Does it appear that the series is convergent or divergent?

$a_n = \frac{n}{1+\sqrt{n}}$. $\quad s_1 = a_1 = \frac{1}{1+\sqrt{1}} = 0.5$

$s_2 = s_1 + a_2 = 0.5 + \frac{2}{1+\sqrt{2}} \approx 1.3284.$

$s_3 = s_2 + a_3 \approx 2.4265$

$s_4 \approx 3.7598 \qquad s_5 \approx 5.3049 \qquad s_6 \approx 7.0443$

$s_7 \approx 8.9644 \qquad s_8 \approx 11.0540.$

It appears that the series is divergent.

Problem #24 Determine whether the geometric series $\sum_{n=0}^{\infty} \frac{1}{(\sqrt{2})^n}$ is convergent or divergent. If it is convergent, find its sum.

This appears to be a geometric series with ratio $r = \frac{1}{\sqrt{2}}$.

Since $|r| = \frac{1}{\sqrt{2}} < 1$, the series converges.

Its sum is $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} = \frac{1}{1-\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2}-1}$

$$= \frac{\sqrt{2}}{\sqrt{2}-1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}+1} = \sqrt{2}\left(\sqrt{2}+1\right) = 2+\sqrt{2}.$$

Problem #40 Determine whether the series $\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n}\right)$ is convergent or divergent. If it is convergent, find its sum.

The series diverges because $\sum_{n=1}^{\infty} \frac{2}{n} = 2\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (If it converged, then $\frac{1}{2} \cdot 2\sum_{n=1}^{\infty} \frac{1}{n}$ would also converge by theorem $8(i)$, but we know from above that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.).

Since the difference of two convergent series is convergent, then if we assume for a moment that $\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n}\right)$ converges, then the difference $\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n}\right) - \sum_{n=1}^{\infty} \frac{3}{5^n}$ must also converge (since $\sum_{n=1}^{\infty} \frac{3}{5^n}$ is a convergent geometric series) and be equal to $\Sigma \frac{2}{n}$.

But we have just seen that $\Sigma \frac{2}{n}$ diverges, so by contradiction we have shown that $\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n}\right)$ must instead diverge.

Problem #48 Determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n^3-n}$ is convergent or divergent by expressing s_n as a telescopic sum. If it is convergent, find its sum.

Using partial fractions, the partial sums of the series $\sum_{n=2}^{\infty} \frac{1}{n^3-n}$ are

$$s_n = \sum_{i=2}^{n} \frac{1}{i(i-1)(i+1)} = \sum_{i=2}^{n} \left(-\frac{1}{i} + \frac{\frac{1}{2}}{i-1} + \frac{\frac{1}{2}}{i+1} \right) = \frac{1}{2} \sum_{i=2}^{n} \left(\frac{1}{i-1} - \frac{2}{i} + \frac{1}{i+1} \right)$$

$$= \frac{1}{2} \left[\left(\frac{1}{1} - \frac{2}{2} + \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6} \right) + \ldots \right.$$

$$\left. + \left(\frac{1}{n-3} - \frac{2}{n-2} + \frac{1}{n-1} \right) + \left(\frac{1}{n-2} + \frac{2}{n-1} + \frac{1}{n} \right) + \left(\frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1} \right) \right]$$

$$= \frac{1}{2} \left(\frac{1}{1} - \frac{2}{2} + \frac{1}{2} + \frac{1}{n} - \frac{2}{n} + \frac{1}{n+1} \right) = \frac{1}{4} - \frac{1}{2n} + \frac{1}{2n+2}.$$

Thus, $\sum_{n=2}^{\infty} \frac{1}{n^3-n} = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \left(\frac{1}{4} - \frac{1}{2n} + \frac{1}{2n+2} \right) = \frac{1}{4}.$

Problem #52 Express the number $0.\overline{46} = 0.46464646\ldots$ as a ratio of integers.

$$0.\overline{46} = \frac{46}{100} + \frac{46}{100^2} + \ldots$$

is a geometric series with $a = \frac{46}{100}$ and $r = \frac{1}{100}$.

It converges to: $\frac{a}{1-r} = \frac{\frac{46}{100}}{1-\frac{1}{100}} = \frac{46}{99}$.

Problem #62 Find the values of x for which the series $\sum_{n=0}^{\infty} \frac{\sin^n x}{3^n}$ converges. Find the sum of the series for those values of x.

$$\sum_{n=0}^{\infty} \left(\frac{\sin x}{3} \right)^n$$

is a geometric series with $r = \frac{\sin x}{3}$,

so the series converges when $|r| < 1$ or $\left| \frac{\sin x}{3} \right| < 1$

$\Rightarrow \quad |\sin x| < 3,$ which is true for all x.

Thus, the sum of the series is $\frac{a}{1-r} = \frac{1}{1-\frac{\sin x}{3}} = \frac{3}{3-\sin x}$.