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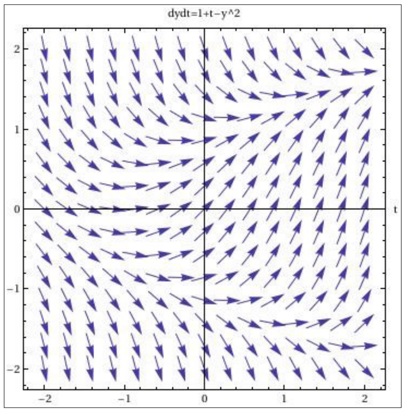
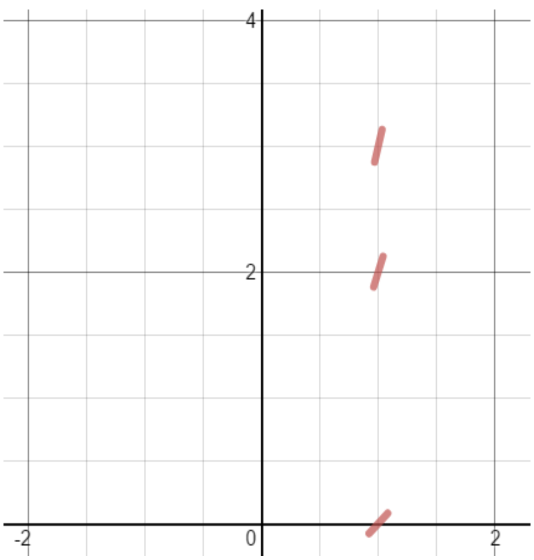
9.2 - Direction Fields and Euler's Method

Review:

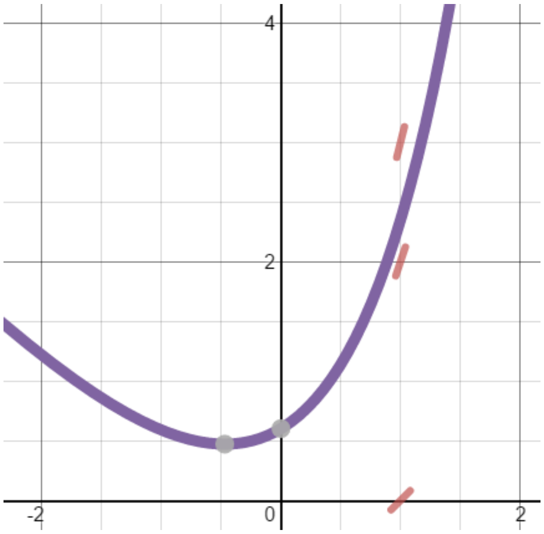
Most differential equations are impossible to solve. However, we can always graph the behavior of the differential equation by creating a direction field.



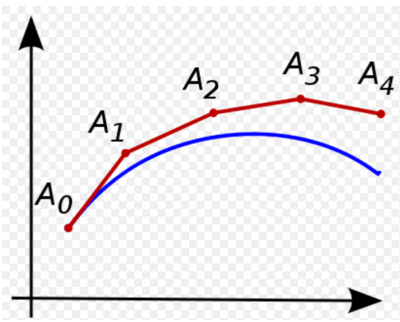
For example, if you were given $y' = x + y$ where $y(0) = 1$.
You could determine the slope in the coordinate plane at each point by choosing a point, let's say $(1,2)$, and then just plugging it into the differential equation. For this case, $y' = 1 + 2 = 3$. So, the slope is 3 at $(1,2)$. Similarly, the slope at $(1,3)$ is 4, and the slope at $(1,1)$ is $2\ldots$



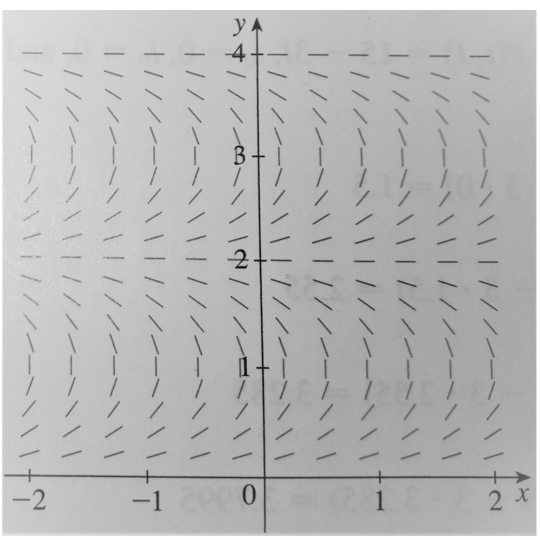
While one could use this method to create a slope field, often computer programs are used to accomplish this. If you were to then graph a particular solution to the initial value problem (as we explored in the previous section of the book), the slope marks you previously made that happen to be near that solution would be parallel it (as seen below).



Euler's Method:
Another way of dealing with an IVP which is not solvable is to approximate values for the solution.
Given: $y' = F(x,y)$, $y(x_0) = y_0$, with step size h, so that $x_n = x_{n-1} + h$.
An approximation is: $y_n = y_{n-1} + h \cdot F(x_{n-1}, y_{n-1})$, with $n = 1, 2, 3, \ldots$
In an attempt to approximate the blue line (solution) below, Euler's method would produce something similar to the red disjointed line (with approximations $A_0 = (x_0, y_0)$, $A_1 = (x_1, y_1)$ etc. The smaller the step size h, the more accurate our approximation will be.



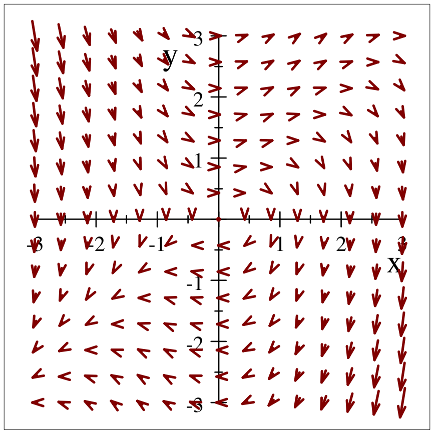
Problem #2 A direction field for the differential equation $y' = \tan\left(\frac{1}{2}\pi y\right)$ is shown.



a) Sketch the graphs of the solutions that satisfy the given initial conditions.
i) $y(0) = 1$, ii) $y(0) = 0.2$,
iii) $y(0) = 2$, iv) $y(1) = 3$.
b) Find all the equilibrium solutions.

It appears that the constant functions $y = 0$, $y = 2$, and $y = 4$ are equilibrium solutions. Note that these three values of y satisfy the given differential equation $y' = \tan\left(\frac{1}{2}\pi y\right)$.

Problem #12 Sketch the direction field of $y' = xy - x^2$. Then use it to sketch a solution curve that passes through the point $(0, 1)$.



Problem #22 Use Euler's method with step size 0.2 to estimate $y(1)$, where $y(x)$ is the solution of the IVP $y' = xy - x^2, \quad y(0) = 1$.

$h = 0.2,\ x_0 = 0,\ y_0 = 1$, and $F(x,y) = xy - x^2$.

Note that $x_1 = x_0 + h = 0 + 0.2 = 0.2,\ x_2 = 0.4,\ x_3 = 0.6,\ x_4 = 0.8$, and $x_5 = 1.0$.

$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.2F(0,1) = 1 + 0.2(0) = 1.$

$y_2 = y_1 + hF(x_1, y_1) = 1 + 0.2F(0.2,1) = 1 + 0.2(0.16) = 1.032.$

$y_3 = 1.032 + 0.2F(0.4, 1.032) = 1.032 + 0.2(0.2528) = 1.08256.$

$y_4 = 1.08256 + 0.2F(0.6, 1.08256) = 1.08256 + 0.2(0.289536) = 1.1404672.$

$y_5 = 1.1404672 + 0.2F(0.8, 1.1404672) = 1.1404672 + 0.2(0.27237376) = 1.194941952.$

Therefore, $y(1) \approx 1.1949$.