Math 5490
Topics in Applied Mathematics
Introduction to the Mathematics of Climate

Fall 2023
1:25 - 3:20 Tuesdays and Thursdays
Amundson Hall 162

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course website
www-users.cse.umn.edu/~mcgehee/teaching/Math5490/

Math 5490
Dynamical Systems

Bifurcation Theory

Hopf Bifurcation

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Bifurcation Theory

$$\frac{dx}{dt} = f(x,\mu), \quad x \in \mathbb{R}^n, \quad \mu \in \mathbb{R}^m$$

state variables — parameters

rest point at $x = 0$ when $\mu = 0$: $f(0,0) = 0$

What happens when we change the parameters?

Poincaré Continuation: Rest point and its classification persist when the Jacobian is nonsingular.

Saddle-node bifurcation: Creation/annihilation of rest points when one eigenvalue is zero.

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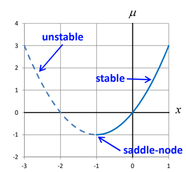
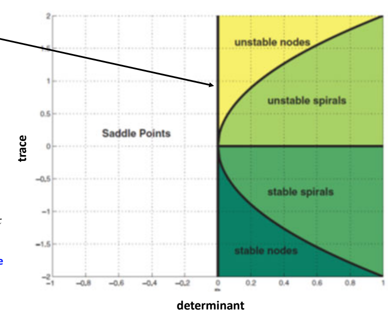
Bifurcation Theory

saddle-node bifurcation

$$\frac{dx}{dt} = f(x,\mu) = \mu - 2x - x^2$$





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For a real matrix, complex eigenvalues always occur as conjugate pairs.

Why?

If λ is an eigenvalue of A with eigenvector v, then

$$Av = \lambda v,$$

which implies that

A is real.

$$A\bar{v} = \bar{A}\bar{v} = \overline{Av} = \overline{\lambda v} = \bar{\lambda}\bar{v}.$$

$$A\bar{v} = \bar{\lambda}\bar{v},$$

so $\bar{\lambda}$ is an eigenvalue of A with eigenvector $\bar{v}$.

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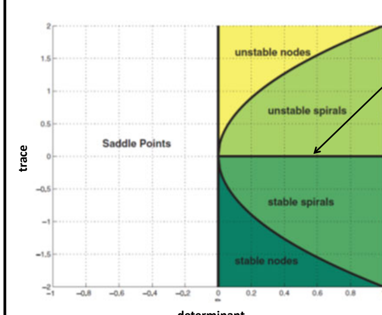
Bifurcation Theory



Hopf Bifurcation

Hopf bifurcation occurs when the real part of a complex eigenvalue passes through zero.

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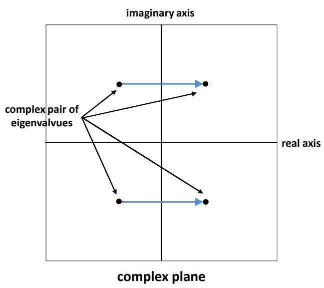
Bifurcation Theory



Hopf bifurcation occurs when the real part of a complex eigenvalue passes through zero as a parameter changes.

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Bifurcation Theory

Example

$$\dot{x} = \mu x - y - (x^2 + y^2)x$$
$$\dot{y} = x + \mu y - (x^2 + y^2)y$$

rest point: $(x, y) = (0,0)$

$$D_1 f((x,y),\mu) = \begin{bmatrix} \mu - 3x^2 - y^2 & -1 - 2xy \\ 1 - 2xy & \mu - 3y^2 - x^2 \end{bmatrix}, \quad D_1 f((0,0),\mu) = \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}$$

characteristic polynomial: $\det \begin{bmatrix} \mu - \lambda & -1 \\ 1 & \mu - \lambda \end{bmatrix} = \lambda^2 - 2\mu\lambda + \mu^2 + 1$

eigenvalues: $\lambda = \dfrac{2\mu \pm \sqrt{4\mu^2 - 4\left(\mu^2 + 1\right)}}{2} = \mu \pm i$

complex pair of eigenvalues crossing the imaginary axis as μ goes from negative to positive.

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Example

$$\dot{x} = \mu x - y - (x^2 + y^2)x$$
$$\dot{y} = x + \mu y - (x^2 + y^2)y$$

Switch to complex coordinates: $z = x + iy$

$$\dot{z} = \dot{x} + i\dot{y} = \mu x - y - (x^2 + y^2)x + i(x + \mu y - (x^2 + y^2)y)$$

$$\dot{z} = ix - y + \mu(x + iy) - (x^2 + y^2)(x + iy)$$

$$\dot{z} = iz + \mu z - |z|^2 z$$

$$\boxed{\dot{z} = (\mu + i)z - |z|^2 z}$$

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Example

$$\dot{z} = (\mu + i)z - |z|^2 z$$

$z = 0$ is a rest point with eigenvalues $\mu \pm i$.

Why?

The variable z is just shorthand for $x+iy$, and we already already computed the eigenvalues.

Alternate Viewpoint.

Turn it into a system by taking the complex conjugate.

$$\frac{dz}{dt} = (\mu + i)z - |z|^2 z$$
$$\frac{d\bar{z}}{dt} = (\mu - i)\bar{z} - |z|^2 \bar{z}$$

$$\frac{d}{dt}\begin{bmatrix} z \\ \bar{z} \end{bmatrix} = \begin{bmatrix} \mu + 1 & 0 \\ 0 & \mu - i \end{bmatrix}\begin{bmatrix} z \\ \bar{z} \end{bmatrix} - |z|^2 \begin{bmatrix} z \\ \bar{z} \end{bmatrix}$$

eigenvalues

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Yet Another Viewpoint

$$\frac{dx}{dt} = \mu x - y - (x^2 + y^2)x$$
$$\frac{dy}{dt} = x + \mu y - (x^2 + y^2)y$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - (x^2 + y^2)\begin{bmatrix} x \\ y \end{bmatrix}$$

coefficient matrix: $A = \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}$ eigenvalues: $\lambda = \mu \pm i$

eigenvalue: $\lambda = \mu + i$ $A - \lambda I = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}$ eigenvector: $\frac{1}{2}\begin{bmatrix} 1 \\ -i \end{bmatrix}$

eigenvalue: $\lambda = \mu - i$ $A - \lambda I = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}$ eigenvector: $\frac{1}{2}\begin{bmatrix} 1 \\ i \end{bmatrix}$

$$S = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \qquad S^{-1}AS = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}\begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} = \begin{bmatrix} \mu + i & 0 \\ 0 & \mu - i \end{bmatrix}$$

We have diagonalized the variational equation.

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Yet Another Viewpoint

$$\dot{x} = \mu x - y - (x^2 + y^2)x$$
$$\dot{y} = x + \mu y - (x^2 + y^2)y$$

$$\begin{matrix} z = x + iy \\ w = x - iy \end{matrix} \Leftrightarrow \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \Leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} z \\ w \end{bmatrix}$$

$$\dot{z} = (\mu + i)z - |z|^2 z$$
$$\dot{w} = (\mu - i)w - |w|^2 w$$

Note that $w = \bar{z}$ and hence that $|w^2| = |\bar{z}^2| = |z^2|$.

The differential equation for w is simply the complex conjugate of the equation for z and hence is redundant.

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Example

$$\dot{z} = (\mu + i)z - |z|^2 z$$

$z = 0$ is a rest point with eigenvalues $\mu \pm i$.

Switch to polar coordinates: $z = re^{i\theta}$

$$\dot{z} = \dot{r}e^{i\theta} + i\dot{\theta}re^{i\theta} = (\mu+i)z - |z|^2 z = (\mu+i)re^{i\theta} - r^2 re^{i\theta}$$

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Example

$$\dot{z} = (\mu + i)z - |z|^2 z$$

$z = 0$ is a rest point with eigenvalues $\mu \pm i$.

Switch to polar coordinates: $z = re^{i\theta}$

$$\dot{z} = \dot{r}e^{i\theta} + i\dot{\theta}re^{i\theta} = (\mu+i)z - |z|^2 z = (\mu+i)re^{i\theta} - r^2 re^{i\theta}$$

$$\dot{r} + i\dot{\theta}r = (\mu+i)r - r^3$$

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Example

$$\dot{z} = (\mu + i)z - |z|^2 z$$

$z = 0$ is a rest point with eigenvalues $\mu \pm i$.

Switch to polar coordinates: $z = re^{i\theta}$

$$\dot{z} = \dot{r}e^{i\theta} + i\dot{\theta}re^{i\theta} = (\mu+i)z - |z|^2 z = (\mu+i)re^{i\theta} - r^2 re^{i\theta}$$

$$\dot{r} + i\dot{\theta}r = (\mu+i)r - r^3$$

$$(\dot{r}) + i(\dot{\theta}r) = (\mu+i)r - r^3 = (\mu r - r^3) + i(r)$$

Equate real and imaginary parts.

$$\dot{r} = \mu r - r^3$$
$$\dot{\theta} = 1$$

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Example

Consider just the radial component.

$$\dot{r} = \mu r - r^3$$

rest points $\quad \mu r - r^3 = r(\mu - r^2) = 0$

$r = 0$ or $r = \sqrt{\mu}$

Jacobian: $\quad D_1 f(r,\mu) = \mu - 3r^2$

rest point at $r = 0$: $\quad D_1 f(0,\mu) = \mu \quad$ stable for $\mu < 0$, unstable for $\mu > 0$

rest point at $r = \sqrt{\mu}$: $\quad D_1 f(\sqrt{\mu},\mu) = -2\mu \quad$ stable for $\mu > 0$

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Example

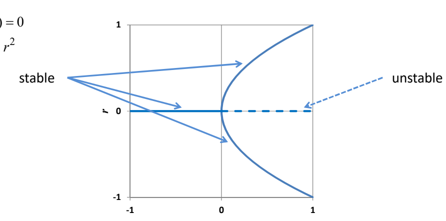
Consider just the radial component.

$$\dot{r} = \mu r - r^3$$

rest points:

$\mu r - r^3 = r(\mu - r^2) = 0$

$r = 0$ and $\mu = r^2$



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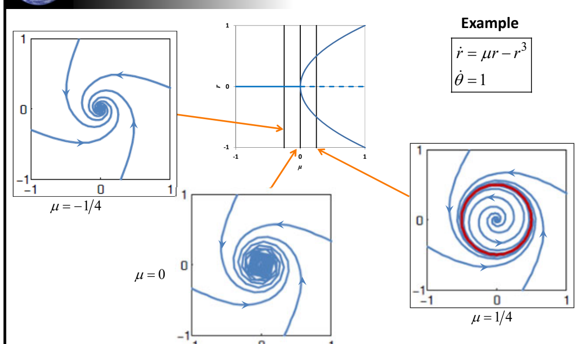
Example

$\dot{r} = \mu r - r^3$

$\dot{\theta} = 1$



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Example

$\dot{z} = (\mu + i)z + |z|^2 z$ $\quad$ $\dot{r} = \mu r + r^3$, $\dot{\theta} = 1$

"+" instead of "-"

rest points $\quad \mu r + r^3 = r(\mu + r^2) = 0$

$r = 0 \quad \text{or} \quad r = \sqrt{-\mu}$

Jacobian: $\quad D_1 f(r,\mu) = \mu + 3r^2$

rest point at $r = 0$: $\quad D_1 f(0,\mu) = \mu \quad$ stable for $\mu < 0$, unstable for $\mu > 0$

rest point at $r = \sqrt{-\mu}$: $\quad D_1 f(\sqrt{-\mu},\mu) = -2\mu \quad$ unstable for $\mu < 0$

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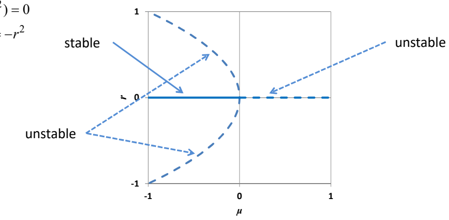
Example

$\dot{r} = \mu r + r^3$

rest points:

$\mu r + r^3 = r(\mu + r^2) = 0$

$r = 0 \quad \text{and} \quad \mu = -r^2$



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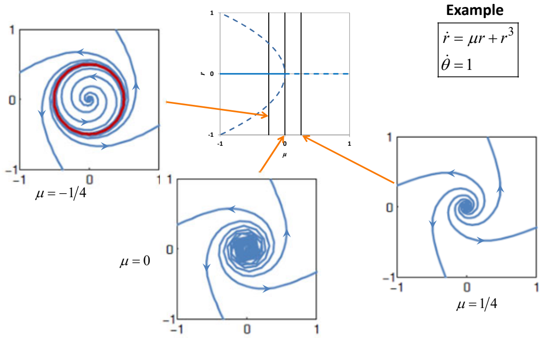
Example

$\dot{r} = \mu r + r^3$
$\dot{\theta} = 1$



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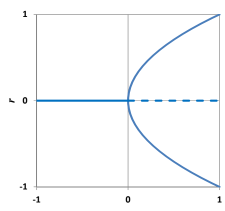
Bifurcation Theory

Hopf Bifurcation

$\dot{z} = (\mu + i)z - |z|^2 z$

supercritical

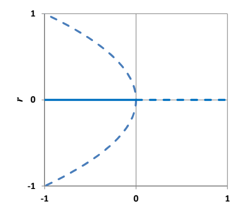
$\dot{r} = \mu r - r^3$
$\dot{\theta} = 1$



$\dot{z} = (\mu + i)z + |z|^2 z$

subcritical

$\dot{r} = \mu r + r^3$
$\dot{\theta} = 1$



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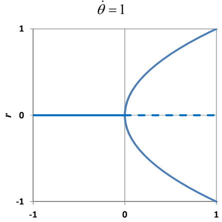
Hopf Bifurcation

$\dot{z} = (\mu + i)z - |z|^2 z$

supercritical

$\dot{r} = \mu r - r^3$
$\dot{\theta} = 1$



narrative

A stable spiral rest point loses stability as a pair of complex eigenvalues pass through the imaginary axis.

A stable periodic orbit is created, growing continuously from the rest point.

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Hopf Bifurcation

narrative

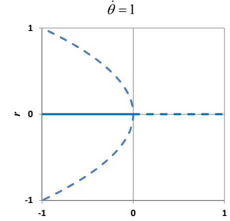
A stable spiral rest point loses stability as a pair of complex eigenvalues pass through the imaginary axis.

A unstable periodic orbit coexists with the stable rest point. As the eigenvalues approach the imaginary axis, the periodic orbit collapses continuously to the rest point and is annihilated by the collision.

$\dot{z} = (\mu + i)z + |z|^2 z$

subcritical

$\dot{r} = \mu r + r^3$
$\dot{\theta} = 1$



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Hopf Bifurcation

narrative continued

As the rest point becomes unstable, the nearby solutions have nowhere to go.

This is another example of a tipping point, where the existing state becomes unstable and the system has to find a new stable state.

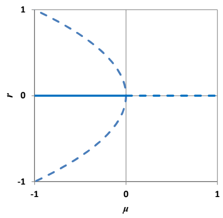
It could find a stable periodic orbit.

$\dot{z} = (\mu + i)z + |z|^2 z$

subcritical

$\dot{r} = \mu r + r^3$

$\dot{\theta} = 1$



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Example

subcritical → new term

$$\dot{z} = (\mu + i)z + |z|^2 z \boxed{- |z|^4 z}$$

$z = 0$ is a rest point with eigenvalues $\mu \pm i$.

Switch to polar coordinates: $z = re^{i\theta}$

$$\dot{z} = \dot{r}e^{i\theta} + i\dot{\theta}re^{i\theta} = (\mu + i)z + |z|^2 z - |z|^4 z = (\mu + i)re^{i\theta} + r^2 re^{i\theta} - r^4 re^{i\theta}$$

$$\dot{r} + i\dot{\theta}r = (\mu + i)r + r^3 - r^5$$

subcritical → new term

$$\dot{r} = \mu r + r^3 \boxed{- r^5}$$

$$\dot{\theta} = 1$$

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Bifurcation Theory

Example

Consider just the radial component.

$$\dot{r} = \mu r + r^3 - r^5$$

rest points: $\mu r + r^3 - r^5 = r(\mu + r^2 - r^4) = 0$

$r = 0$ and $\mu = -r^2 + r^4$

Jacobian: $D_1 f(r, \mu) = \mu + 3r^2 - 5r^4$

rest point at $r = 0$: $D_1 f(0, \mu) = \mu$ stable for $\mu < 0$, unstable for $\mu > 0$

rest point when $\mu + r^2 - r^4 = 0$: $D_1 f(r, \mu) = \mu + 3r^2 - 5r^4 = -r^2 + r^4 + 3r^2 - 5r^4 = 2r^2 - 4r^4 = 2r^2(1 - 2r^2)$

unstable when $r^2 < 1/2$, stable when $r^2 > 1/2$

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Bifurcation Theory

Example

$\dot{z} = (\mu + i)z + |z|^2 z - |z|^4 z$

radial component

$\dot{r} = \mu r + r^3 - r^5$

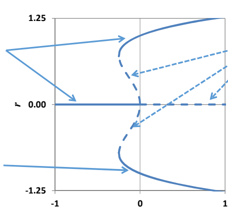
rest points:

$\mu r + r^3 - r^5 = r(\mu + r^2 - r^4) = 0$

$r = 0$ and $\mu = -r^2 + r^4$

rest point at $r = 0$:
stable for $\mu < 0$,
unstable for $\mu > 0$

rest point when
$\mu + r^2 - r^4 = 0$:
unstable when $r^2 < 1/2$
stable for when $r^2 > 1/2$



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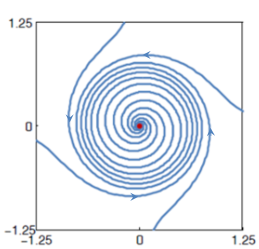
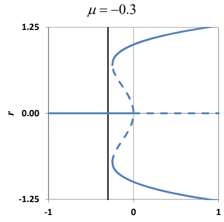
Bifurcation Theory

Example

$\dot{z} = (\mu + i)z + |z|^2 z - |z|^4 z$



$\mu = -0.3$



For $\mu < 0.25$, there is a single stable rest point.

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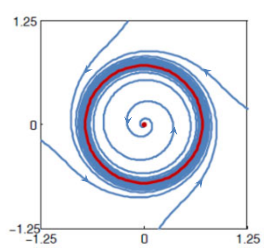
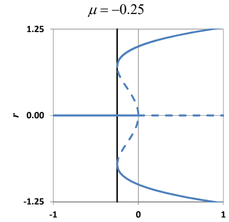
Example

$\dot{z} = (\mu + i)z + |z|^2 z - |z|^4 z$



$\mu = -0.25$



At $\mu = -0.25$, a periodic orbit is created, far from the rest point but surrounding it.

The rest point remains stable.

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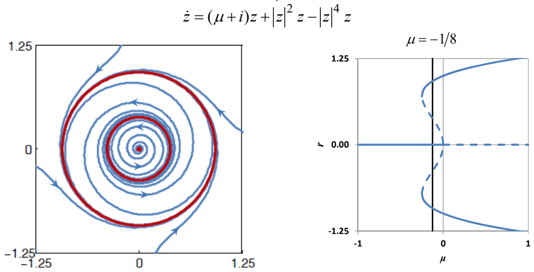
Bifurcation Theory

Example

$$\dot{z} = (\mu + i)z + |z|^2 z - |z|^4 z$$



For $-0.25 < \mu < 0$, the periodic orbit has split into a stable periodic orbit and an unstable one. The stable orbit grows larger, while the unstable one shrinks toward the stable rest point.

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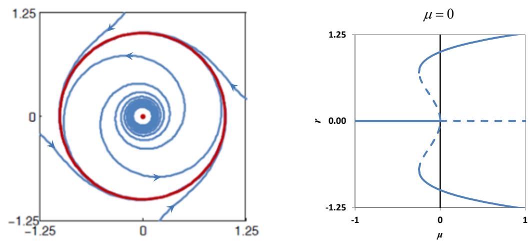
Example

$$\dot{z} = (\mu + i)z + |z|^2 z - |z|^4 z$$



At $\mu = 0$, the unstable periodic orbit has collapsed to the origin, annihilating itself and causing the rest point to lose its stability and undergo a subcritical Hopf bifurcation.

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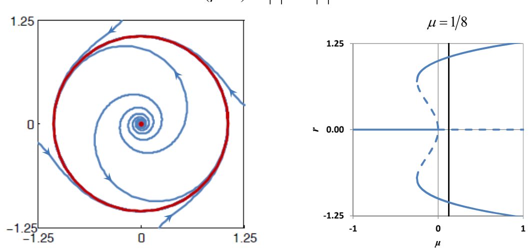
Example

$$\dot{z} = (\mu + i)z + |z|^2 z - |z|^4 z$$



For $\mu > 0$, the rest point has become an unstable spiral, while the stable periodic orbit continues to grow.

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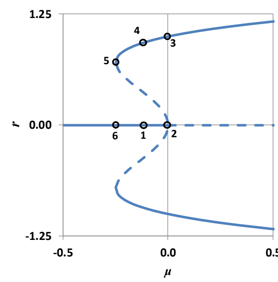
Example

$$\dot{z} = (\mu + i)z + |z|^2 z - |z|^4 z$$



Hysteresis

1. Start at rest point when $\mu = -0.125$.
2. Increase μ to 0. Subcritical Hopf occurs.
3. System jumps to stable periodic orbit.
4. Return μ to original value. Still on stable periodic orbit.
5. Decrease μ to -0.25. Stable periodic orbit and unstable one annihilate each other.
6. System jumps back to the stable rest point.
7. Increase μ to original value.

Math 5490
Bifurcation Theory

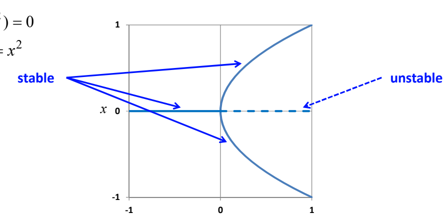
Pitchfork Bifurcation

$$\dot{x} = \mu x - x^3$$

rest points: (Same as the r equation for Hopf)

$\mu x - x^3 = x(\mu - x^2) = 0$

$x = 0$ and $\mu = x^2$



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Pitchfork Bifurcation

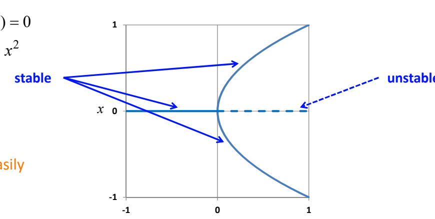
$$\dot{x} = \mu x - x^3$$

rest points: (Same as the r equation for Hopf)

$\mu x - x^3 = x(\mu - x^2) = 0$

$x = 0$ and $\mu = x^2$

Problem:
This picture is easily destroyed.



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Pitchfork Bifurcation

$$\dot{x} = \mu x - x^3$$

Add another parameter.

$$\dot{x} = \lambda + \mu x - x^3$$

rest points: $\lambda + \mu x - x^3 = 0, \quad \mu = x^2 - \frac{\lambda}{x}$

Jacobian: $D_1 f(x,\mu,\lambda) = \mu - 3x^2$

stable if $\mu < 3x^2$, unstable if $\mu > 3x^2$

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Bifurcation Theory

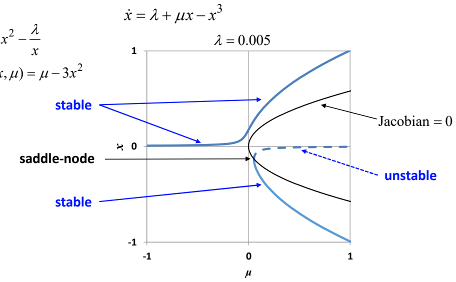
Pitchfork Bifurcation

Add another parameter.

$$\dot{x} = \lambda + \mu x - x^3$$

rest points: $\mu = x^2 - \frac{\lambda}{x}$

Jacobian: $D_1 f(x,\mu) = \mu - 3x^2$

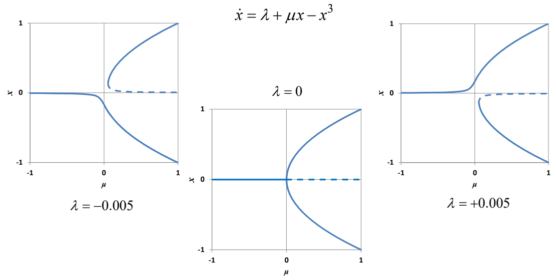


Math 5490
Bifurcation Theory

Pitchfork Bifurcation

Add another parameter.

$$\dot{x} = \lambda + \mu x - x^3$$



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Bifurcation Theory

Hopf bifurcation is an interesting and well-studied phenomenon that arises in many applications.

Curiously, it is difficult to find examples in climate models.

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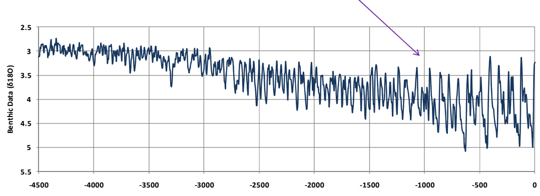
Bifurcation Theory

Is the mid Pleistocene transition a Hopf bifurcation?



Lisiecki, L. E., and M. E. Raymo (2005), A Pliocene-Pleistocene stack of 57 globally distributed benthic d18O records, *Paleoceanography* **20**, PA1003, doi:10.1029/2004PA001071.

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Bifurcation Theory

Salzman-Maasch Model

Milankovitch forcing

global ice mass $\longrightarrow$ $\dot{X} = -X - Y - uM(t)$

atmospheric CO_2 $\longrightarrow$ $\dot{Y} = -pZ + rY + sZ^2 - Z^2Y$

ocean circulation $\longrightarrow$ $\dot{Z} = -q(X + Z)$

Barry Salzman and Kirk A. Maasch, "A Low-Order Dynamical Model of Global Climatic Variability Over the Full Pleistocene," *Journal of Geophysical Research* **95** (D2), 1955-1963 (1990)

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Bifurcation Theory

Salzman-Maasch Model

unforced (Milankovitch turned off)

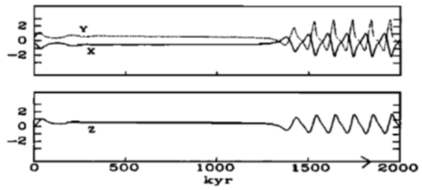

Fig. 7. Unforced (free) solution for (4)–(6) with $q = 1.2$, $s = 0.8$, and p and r varying linearly between $0.8 \to 1.0$ and $0.7 \to 0.8$, respectively. (Top) Model solutions for X (solid) and Y (dashed). (Bottom) Model solution for Z.

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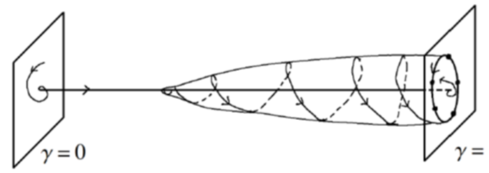
Bifurcation Theory

Dynamic Hopf Bifurcation



Samantha Oestreicher, *Forced Oscillators with Dynamic Hopf Bifurcations and applications to Paleoclimate,* PhD Thesis, University of Minnesota 2014.

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Bifurcation Theory

Salzman-Maasch Model

The Salzman-Maasch model shows how the carbon cycle and the ocean currents can interact to produce unforced oscillations with periods of about 100,000 years. The same model with slightly different parameters can exhibit stationary behavior.

Looks a lot like a Hopf bifurcation.

Problem:

It is just a coincidence that the intrinsic period of the glacial cycles happens to coincide with the period of the eccentricity cycles.

What happens if we force the system with Milankovitch?

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Bifurcation Theory

Salzman-Maasch Model

Milankovitch forcing

global ice mass $\longrightarrow$ $\dot{X} = -X - Y - uM(t)$

atmospheric CO_2 $\longrightarrow$ $\dot{Y} = -pZ + rY + sZ^2 - Z^2 Y$

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Barry Salzman and Kirk A. Maasch, "A Low-Order Dynamical Model of Global Climatic Variability Over the Full Pleistocene," *Journal of Geophysical Research* **95** (D2), 1955-1963 (1990)

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Bifurcation Theory

Salzman-Maasch Model

forced

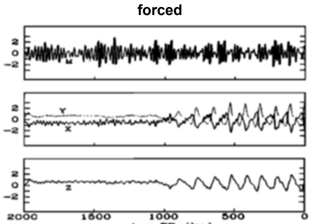

Fig. 8. Forced solution for (4)–(6) with $q = 1.2$, $s = 0.8$, $u = 0.7$, and p and r varying linearly between $0.8 \to 1.0$ and $0.7 \to 0.8$, respectively. (Top) Normalized 65°N July insolation curve (M) used as forcing. (Middle) Model solutions for X (solid) and Y (dashed). (Bottom) Model solution for Z.

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Bifurcation Theory

Is the mid Pleistocene transition a Hopf bifurcation?

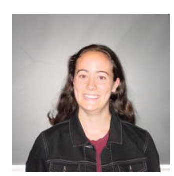
Additional Coincidence:

As observed by Hays et al, the glacial cycles are in phase with the eccentricity and the obliquity.

Oestreicher showed that forced dynamic Hopf bifurcations exhibit unpredictable phase shifts. It is another "cosmic coincidence" that the phase of the glacial cycles line up with the Milankovitch cycles.

Conclusion:

Hopf bifurcation might not be the best explanation for the glacial cycles.



Samantha Schumacher (née Oestreicher)

Samantha Oestreicher, *Forced Oscillators with Dynamic Hopf Bifurcations and applications to Paleoclimate,* PhD Thesis, University of Minnesota 2014.

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Math 5490
Bifurcation Theory

What caused the Dansgaard-Oeschger oscillations?

They could be self-oscillations in the natural dynamics of ocean circulation.

Welander constructed a simple (***conceptual!***) box model of ocean circulation and showed that the interactions of temperature and salinity with the atmosphere, the surface ocean, and the deep ocean could create self-oscillations.

R/V WeeLander is a 23-foot-long Beach Master work boat, informally named in honor of Professor Pierre Welander (1925—1996).

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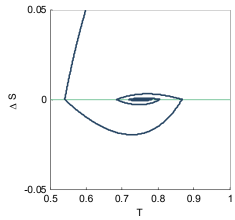
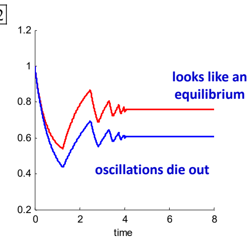
Math 5490
Ocean Circulation

Welander's Model

$$\dot{T} = 1 - T - k(\rho)T \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases} \qquad \alpha = 0.8$$
$$\dot{S} = \beta(1 - S) - k(\rho)S \qquad \beta = 0.5$$
$$\rho = -\alpha T + S$$

$\varepsilon = 0.002$





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Math 5490
Ocean Circulation

Welander's Model

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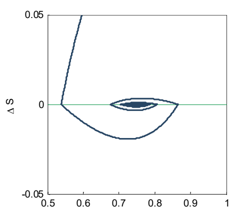
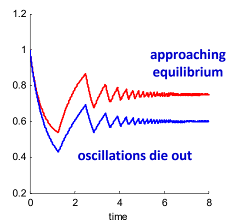
$$\rho = -\alpha T + S$$

$\varepsilon = 0$





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Math 5490
Ocean Circulation

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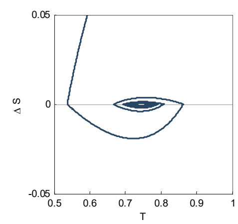
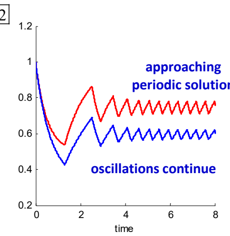
$$\rho = -\alpha T + S$$

$\varepsilon = -0.002$





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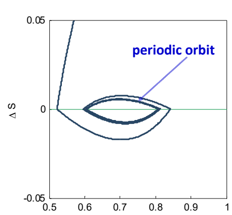
Ocean Circulation

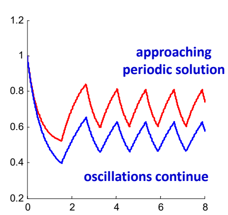
Welander's Model

$\varepsilon = -0.02$

$$\dot{T} = 1 - T - k(\rho)T \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases} \qquad \alpha = 0.8$$
$$\dot{S} = \beta(1 - S) - k(\rho)S \qquad \beta = 0.5$$
$$\rho = -\alpha T + S$$

$\varepsilon = -0.02$





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Math 5490
Ocean Circulation

Welander's Model

Mathematical Note

Welander's model has the features of a Hopf bifurcation. Something behaving like a stable rest point becomes unstable and spins off a stable periodic orbit.

The mathematical tools to actually *prove* that Welander's model behaves in the way he described were not fully developed in 1982.

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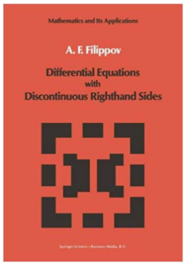
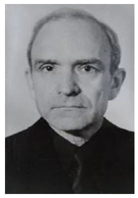
Ocean Circulation



Kluwer Academic Publishers

Published in Russian in 1985.



A.F. Filippov*

*https://alchetron.com/Aleksei-Fedorovich-Filippov

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Ocean Circulation

Mathematical Note

Welander assumed that the self-oscillations he found in his discontinuous model would hold held for a nearby smooth system.

Juliann Leifeld, PhD 2016:

Welander's assumption was correct.