Math 5490
Topics in Applied Mathematics
Introduction to the Mathematics of Climate

Fall 2023
1:25 - 3:20 Tuesdays and Thursdays
Amundson Hall 162

Richard McGehee, Instructor
458 Vincent Hall
mcgehee@umn.edu
www-users.cse.umn.edu/~mcgehee/

course website
www-users.cse.umn.edu/~mcgehee/teaching/Math5490/

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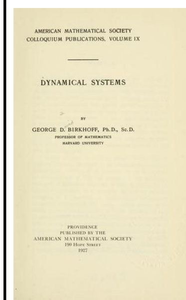
Math 5490
Dynamical Systems



https://openlibrary.org/works/OL86546W/Dynamical_systems

https://en.wikipedia.org/wiki/George_David_Birkhoff

https://en.wikipedia.org/wiki/Henri_Poincar%C3%A9

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Math 5490
Dynamical Systems

Nonlinear Systems

One Variable

$$\frac{dx}{dt} = f(x)$$

Rest points: $\frac{dx}{dt} = 0 \iff f(x) = 0$

If $f(p) = 0$, then $x(t) = p$ (constant) is a solution.

Example

$$\frac{dx}{dt} = x - x^3$$

Rest points: $x - x^3 = 0 \iff x(1-x)(1+x) = 0$

$-1,\ 0,\ \text{and}\ 1$

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Discussion

What are the rest points of this equation?

$$\frac{dx}{dt} = x^2 - 1$$

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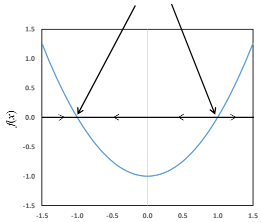
Discussion

What are the restpoints of this equation?

$$\frac{dx}{dt} = x^2 - 1$$

$$x^2 - 1 = 0$$
$$x = \pm 1$$



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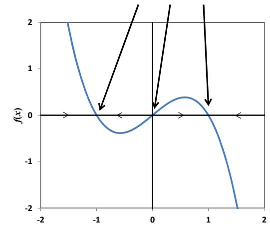
One Variable

Example

$$\frac{dx}{dt} = x - x^3$$

Rest points: $-1,\ 0,\ \text{and}\ 1$



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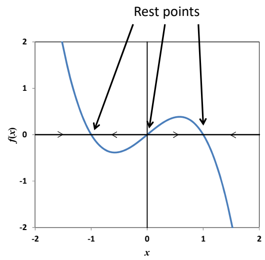
One Variable

Example

$$\frac{dx}{dt} = x - x^3$$

Rest points: $-1,\ 0,\ \text{and}\ 1$

What about the stability of the rest points?



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Nonlinear Systems

$$\frac{dx}{dt} = f(x) \qquad \text{Rest point } p: \quad f(p) = 0$$

Linear Approximation

$$f(x) \approx f(p) + f'(p)(x-p) = f'(p)(x-p)$$

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Nonlinear Systems

$$\frac{dx}{dt} = f(x) \qquad \text{Rest point } p: \quad f(p) = 0$$

Linear Approximation

$$f(x) \approx f(p) + f'(p)(x-p) = f'(p)(x-p)$$

Introduce $\xi = x - p$.

Then $f(x) = f(p+\xi) \approx f'(p)\xi$

$$\frac{d\xi}{dt} = \frac{dx}{dt} = f(x) = f(p+\xi) \approx f'(p)\xi$$

$$\frac{d\xi}{dt} = f'(p)\xi$$

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$$\frac{d\xi}{dt} = \frac{dx}{dt} = f(x) = f(p+\xi) \approx f'(p)\xi$$

$$\frac{d\xi}{dt} = f'(p)\xi$$

Basic Idea

If ξ is small, i.e., if x is close to p, then solutions of $\frac{d\xi}{dt} = f(\xi + p)$ are close to solutions of $\frac{d\xi}{dt} = f'(p)\xi$.

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Nonlinear Systems

$$\frac{dx}{dt} = f(x) \qquad \text{Rest point } p: \quad f(p) = 0$$

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Basic Idea

If ξ is small, i.e., if x is close to p, then solutions of $\frac{d\xi}{dt} = f(\xi + p)$ are close to solutions of $\frac{d\xi}{dt} = f'(p)\xi$.

The rest point p is asymptotically stable for $\frac{dx}{dt} = f(x)$ if the origin is asymptotically stable for $\frac{d\xi}{dt} = f'(p)\xi$. ← **linear, one variable** $\frac{d\xi}{dt} = a\xi$

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$$\frac{dx}{dt} = f(x) \qquad \text{Rest point } p: \quad f(p) = 0$$

Variational Equation

$$\frac{d\xi}{dt} = f'(p)\xi$$

$$\frac{d\xi}{dt} = a\xi, \quad \text{stable if } a < 0, \text{ unstable if } a > 0$$

$$a = f'(p)$$

Stability Criteria

The rest point p is asymptotically stable for $\frac{dx}{dt} = f(x)$ if $f'(p) < 0$.

It is unstable if $f'(p) > 0$.

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Example

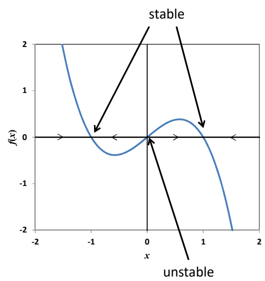
$$\frac{dx}{dt} = f(x) = x - x^3$$

Rest points: $-1,\ 0,$ and 1

$f'(x) = 1 - 3x^2$

$f'(-1) = f'(1) = -2, \quad f'(0) = 1$

Rest points -1 and 1 are stable, rest point 0 is unstable.



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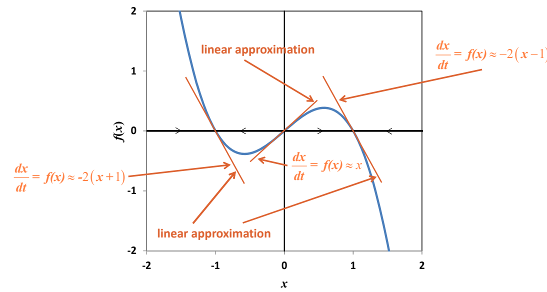
Example



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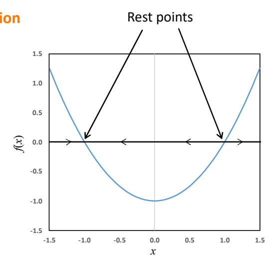
Discussion

$$\frac{dx}{dt} = x^2 - 1$$

restpoints: $x^2 - 1 = 0 \quad x = \pm 1$

What are the associated variational equations for each of these points?



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Discussion

$$\frac{dx}{dt} = x^2 - 1$$

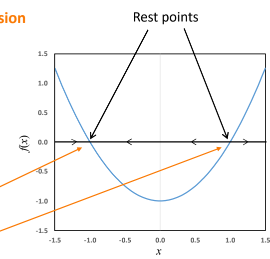
restpoints: $x^2 - 1 = 0 \quad x = \pm 1$

What are the associated variational equations for each of these points?

$f(x) = x^2 - 1 \qquad f'(x) = 2x$

$p = -1: \quad f'(-1) = -2 \quad \boxed{\frac{d\xi}{dt} = -2\xi}$

$p = +1: \quad f'(1) = 2 \quad \boxed{\frac{d\xi}{dt} = 2\xi}$



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Two Variables

$$\frac{dx_1}{dt} = f_1(x_1, x_2) \qquad \frac{dx_2}{dt} = f_2(x_1, x_2)$$

$$\frac{dx}{dt} = \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = f(x)$$

$$\boxed{\frac{dx}{dt} = f(x), \quad x \in \mathbb{R}^2, \quad f : \mathbb{R}^2 \to \mathbb{R}^2}$$

$$\frac{dx}{dt} = 0 \quad \Leftrightarrow \quad f(x) = 0$$

Rest Points

If $f(p) = 0$, then $x(t) = p$ (constant) is a solution (rest point)

$$f(p) = 0 \quad \Leftrightarrow \quad \begin{matrix} f_1(p_1, p_2) = 0 \\ f_1(p_1, p_2) = 0 \end{matrix} \qquad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = p$$

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Nonlinear Two Variable Systems

Example

$$\frac{dx}{dt} = x - x^3 + y$$

$$\frac{dy}{dt} = -y$$

Rest Points

$$\begin{matrix} x - x^3 + y = 0 \\ -y = 0 \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} x - x^3 = 0 \\ y = 0 \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} x(1-x)(1+x) = 0 \\ y = 0 \end{matrix}$$

$$\boxed{(-1,0),\ (0,0),\ \text{and}\ (1,0)}$$

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Example

$$\frac{dx}{dt} = x - x^3 + y$$
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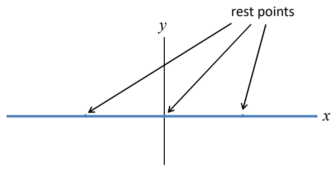

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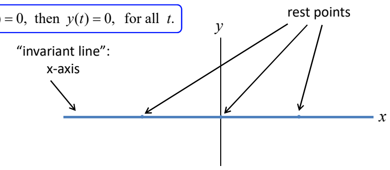
Nonlinear Two Variable Systems

Example

$$\frac{dx}{dt} = x - x^3 + y$$
$$\frac{dy}{dt} = -y$$

If $y(0) = 0$, then $y(t) = 0$, for all t.



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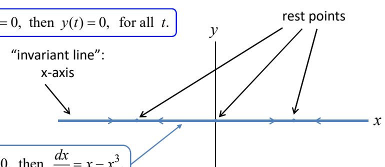

If $y(0) = 0$, then $\frac{dx}{dt} = x - x^3$.

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Nonlinear Systems

Discussion

What are the rest points of this system?

$$\frac{dx}{dt} = x^2 - 1 + y$$
$$\frac{dy}{dt} = -y$$

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Discussion

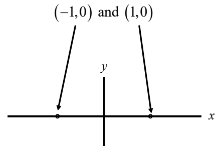
What are the rest points of this system?

$$\frac{dx}{dt} = x^2 - 1 + y$$
$$\frac{dy}{dt} = -y$$

$x^2 - 1 + y = 0 \quad x = \pm 1$

$-y = 0 \quad y = 0$

$(-1,0)$ and $(1,0)$



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Nonlinear Two Variable Systems

Example

$$\frac{dx}{dt} = x - x^3 + y$$
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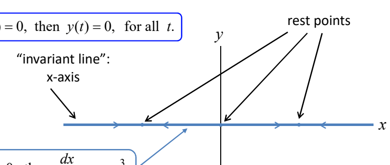
If $y(0) = 0$, then $y(t) = 0$, for all t.



If $y(0) = 0$, then $\frac{dx}{dt} = x - x^3$.

How do we analyze the full system?

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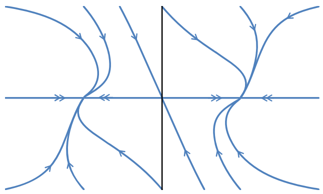
Example

$$\frac{dx}{dt} = x - x^3 + y$$
$$\frac{dy}{dt} = -y$$

Preview



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Nonlinear Two Variable Systems

Jacobian Matrix

$$\frac{dx_1}{dt} = f_1(x_1, x_2)$$
$$\frac{dx_2}{dt} = f_2(x_1, x_2)$$

$$\frac{dx}{dt} = f(x), \quad x \in \mathbb{R}^2, \quad f : \mathbb{R}^2 \to \mathbb{R}^2$$

$$Df(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_1, x_2) & \frac{\partial f_1}{\partial x_2}(x_1, x_2) \\ \frac{\partial f_2}{\partial x_1}(x_1, x_2) & \frac{\partial f_2}{\partial x_2}(x_1, x_2) \end{bmatrix}$$

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Jacobian Matrix

Example

$$\frac{dx}{dt} = x - x^3 + y$$
$$\frac{dy}{dt} = -y$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} = \begin{bmatrix} x - x^3 + y \\ -y \end{bmatrix}$$

$$Df\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \end{bmatrix} = \begin{bmatrix} 1 - 3x^2 & 1 \\ 0 & -1 \end{bmatrix}$$

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Nonlinear Two Variable Systems

Linear Approximation

one independent variable $\quad f(x) \approx f(p) + f'(p)(x - p)$

two independent variables $\quad f(x,y) \approx f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$

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Linear Approximation

one independent variable $\quad f(x) \approx f(p) + f'(p)(x - p)$

two independent variables $\quad f(x,y) \approx f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$

Example

$$f(x,y) = x - x^3 + y$$

$$\frac{\partial f}{\partial x} f(x,y) = 1 - 3x^2 \qquad \frac{\partial f}{\partial y} f(x,y) = 1 - 3x^2$$

$$f(x,y) \approx f(x_0, y_0) + (1 - 3x_0^2)(x - x_0) + (y - y_0)$$

$$(x_0, y_0) = (0,0) \quad \Rightarrow \quad f(x,y) \approx f(0,0) + (1 - 3 \times 0^2)(x - 0) + (y - 0) = x + y$$

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Linear Approximation

two by two system

derivative

one variable $\longrightarrow f(x) \approx f(p) + f'(p)(x - p) = f'(p)(x - p)$

Jacobian

two variables $\longrightarrow f(x) \approx f(p) + Df(p)(x - p) = Df(p)(x - p)$

$$\begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} \approx \begin{bmatrix} f_1(p_1, p_2) \\ f_2(p_1, p_2) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(p_1, p_2) & \frac{\partial f_1}{\partial x_2}(p_1, p_2) \\ \frac{\partial f_2}{\partial x_1}(p_1, p_2) & \frac{\partial f_2}{\partial x_2}(p_1, p_2) \end{bmatrix} \begin{bmatrix} x_1 - p_1 \\ x_2 - p_2 \end{bmatrix}$$

$$f_1(x_1, x_2) \approx f_1(p_1, p_2) + \frac{\partial f_1}{\partial x_1}(p_1, p_2)(x_1 - p_1) + \frac{\partial f_1}{\partial x_2}(p_1, p_2)(x_2 - p_2)$$

$$f_2(x_1, x_2) \approx f_2(p_1, p_2) + \frac{\partial f_2}{\partial x_1}(p_1, p_2)(x_1 - p_1) + \frac{\partial f_2}{\partial x_2}(p_1, p_2)(x_2 - p_2)$$

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Nonlinear Two Variable Systems

Example

$$\frac{dx}{dt} = x - x^3 + y$$
$$\frac{dy}{dt} = -y$$

Rest Points: $(x,y) = (-1,0),\ (0,0),\ \text{and}\ (1,0)$

Jacobian: $Df(x,y) = \begin{bmatrix} 1-3x^2 & 1 \\ 0 & -1 \end{bmatrix}$

$$Df(-1,0) = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \quad Df(0,0) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \quad Df(1,0) = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$$

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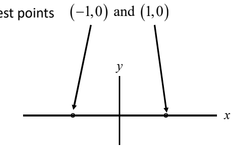
Discussion

$$\frac{dx}{dt} = x^2 - 1 + y$$
$$\frac{dy}{dt} = -y$$

rest points $(-1,0)$ and $(1,0)$



What are the Jacobian matrices for each of these points?

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$$\frac{dx}{dt} = x^2 - 1 + y$$
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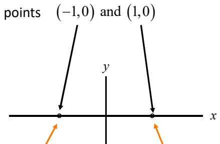

What are the Jacobian matrices for each of these points?

$$Df(x,y) = \begin{bmatrix} 2x & 1 \\ 0 & -1 \end{bmatrix} \quad Df(-1,0) = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \quad Df(1,0) = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$$

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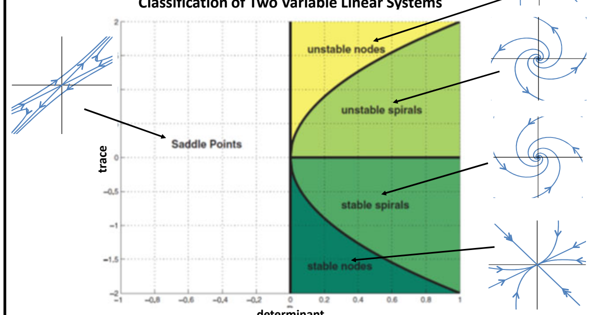
Classification of Two Variable Linear Systems



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Nonlinear Two Variable Systems

Example

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$$Df(-1,0) = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \quad Df(0,0) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \quad Df(1,0) = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$$

$Df(-1,0)$	$Df(0,0)$	$Df(1,0)$
$\delta = \det = 2 > 0$	$\delta = \det = -1 < 0$	$\delta = \det = 2 > 0$
$\tau = \text{trace} = -3 < 0$		$\tau = \text{trace} = -3 < 0$
$\tau^2 - 4\delta = 1 > 0$	saddle	$\tau^2 - 4\delta = 1 > 0$
sink (stable node)		sink (stable node)

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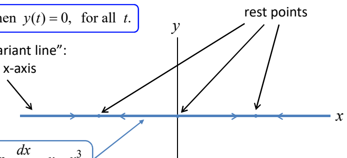
Nonlinear Two Variable Systems

Example

$$\frac{dx}{dt} = x - x^3 + y$$
$$\frac{dy}{dt} = -y$$

If $y(0) = 0$, then $y(t) = 0$, for all t.

"invariant line": x-axis



If $y(0) = 0$, then $\frac{dx}{dt} = x - x^3$.

How do we analyze the full system?

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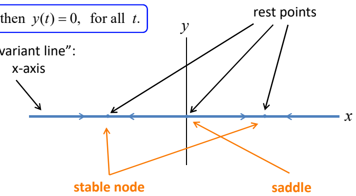

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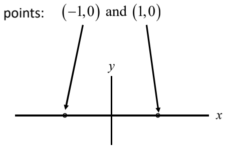
rest points: $(-1,0)$ and $(1,0)$



Classify each of the rest points.

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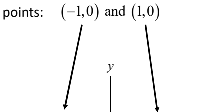
rest points: $(-1,0)$ and $(1,0)$



Classify each of the rest points.

$$\begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$$

eigenvalues: -2 and -1 $\qquad$ 2 and -1

stable node $\qquad$ saddle

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Dynamical Systems

Nonlinear Two Variable Systems

What else can we learn from the variational equation?

$$\frac{dx}{dt} = f(x) \qquad \text{Rest point } p: \; f(p) = 0$$

Linear approximation:

$$f(x) \approx f(p) + Df(p)(x-p) = Df(p)(x-p)$$

Introduce $\xi = x - p$.

Then $f(x) = f(p+\xi) \approx Df(p)\xi$

$$\frac{d\xi}{dt} = \frac{dx}{dt} = f(x) = f(p+\xi) \approx Df(p)\xi$$

Basic Idea

If ξ is small, i.e., if x is close to p, then solutions of $\frac{d\xi}{dt} = f(\xi + p)$ are close to solutions of $\frac{d\xi}{dt} = Df(p)\xi$.

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Example

$$\frac{dx}{dt} = x - x^3 + y$$
$$\frac{dy}{dt} = -y$$

Rest Points: $(x,y) = (-1,0),\ (0,0),$ and $(1,0)$

Jacobian: $Df(x,y) = \begin{bmatrix} 1-3x^2 & 1 \\ 0 & -1 \end{bmatrix}$

$$Df(-1,0) = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \quad Df(0,0) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \quad Df(1,0) = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$$

sink (stable node) $\qquad$ saddle $\qquad$ sink (stable node)

eigenvalues: $1, \; -1$

eigenvectors: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

unstable $\qquad$ stable

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Jacobian:

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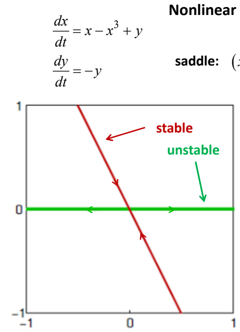

variational equation

$$\dot{\xi} = A\xi, \quad A = Df(0,0) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

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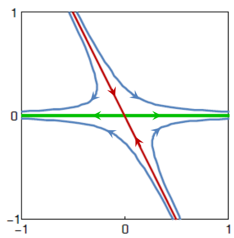

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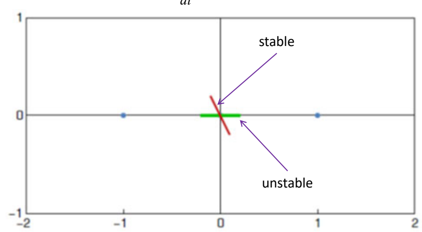

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fast slow

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Example

$$\frac{dx}{dt} = x - x^3 + y$$
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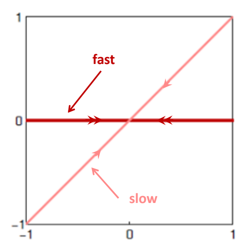

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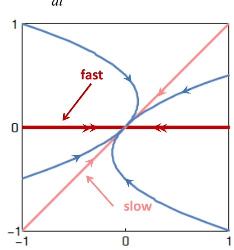

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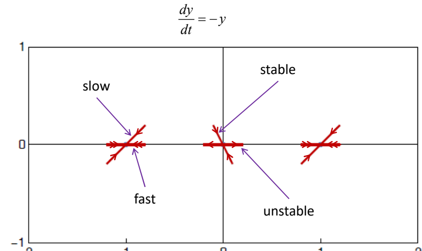

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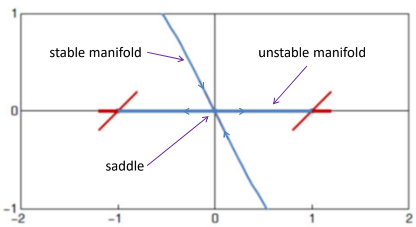

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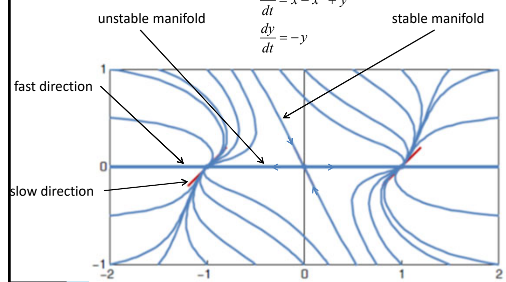

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Discussion

$$\frac{dx}{dt} = x^2 - 1 + y$$
$$\frac{dy}{dt} = -y$$

Find the eigenvectors for each of the rest points.

Sketch the solutions of the variational equation for each of the rest points.

rest points: $(-1,0)$ $(1,0)$ (on the x-axis, with y-axis shown)

$\begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$ $\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$

eigenvalues: -2 and -1 $\quad$ 2 and -1

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eigenvalues: -2 and -1 $\quad$ 2 and -1

eigenvectors: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$