Math 5490
Topics in Applied Mathematics
Introduction to the Mathematics of Climate

Fall 2023
1:25 - 3:20 Tuesdays and Thursdays
Amundson Hall 162

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course website
www-users.cse.umn.edu/~mcgehee/teaching/Math5490/

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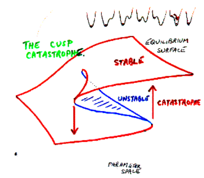
Math 5490
Dynamical Systems

Catastrophe Theory

Sir Christopher Zeeman
1925-2016
https://en.wikipedia.org/wiki/Christopher_Zeeman



https://annex.exploratorium.edu/complexity/CompLexicon/catastrophe.html

Rene Thom, 1923-2002
https://en.wikipedia.org/wiki/Ren%C3%A9_Thom

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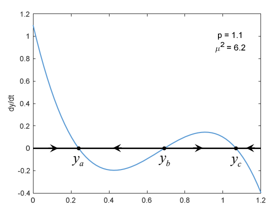
Tipping Points

Cessi's Model

$$\frac{dy}{dt} = -\left(1+\mu^2\left(y-1\right)^2\right)y + p$$



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Tipping Points

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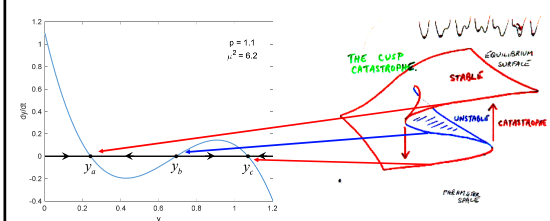

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Tipping Points

Cessi's Model

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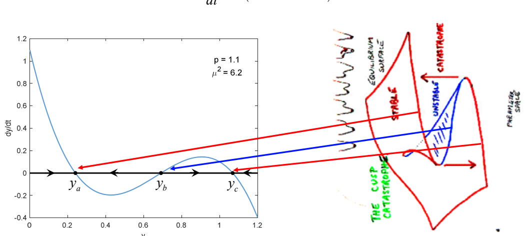

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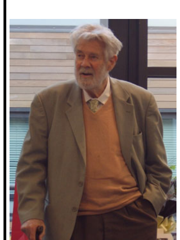

Sir Christopher Zeeman
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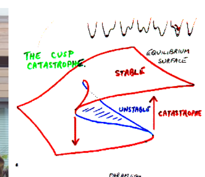

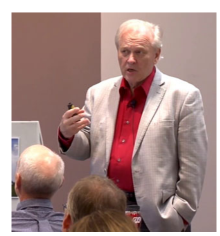

Clarence Lehman

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Linear Systems Summary

$$\frac{dx}{dt} = Ax$$

For linear systems of ordinary differential equations, the eigenvalues and eigenvectors tell us a lot.

They tell us solutions.

If λ is an eigenvalue with corresponding eigenvector v, then $x(t) = e^{\lambda t}v$ is a solution.

They tell us simplifying transformations.

If λ_1 and λ_2 are eigenvalues of A with corresponding eigenvectors v_1 and v_2, if $S = [v_1 \mid v_2]$, and if $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, then $S^{-1}AS = \Lambda$.

Linear Systems Summary

$$\frac{dx}{dt} = Ax$$

For linear systems of ordinary differential equations, the eigenvalues and eigenvectors tell us a lot.

Reference

Kaper & Engler, *Mathematics & Climate*, SIAM 2013, Chapter 4

Coordinate Change

$$\frac{dx}{dt} = Ax$$

If λ_1 and λ_2 are eigenvalues of A with corresponding eigenvectors v_1 and v_2,

if $S = [v_1 \mid v_2]$, and if $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, then $S^{-1}AS = \Lambda$.

$$x = Sy \;\Rightarrow\; S\frac{dy}{dt} = \frac{dx}{dt} = ASy \;\Rightarrow\; \frac{dy}{dt} = S^{-1}ASy = \Lambda y$$

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2, \quad \frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2$$

becomes

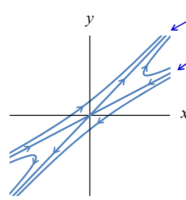
$$\frac{dy_1}{dt} = \lambda_1 y_1, \quad \frac{dy_2}{dt} = \lambda_2 y_2$$

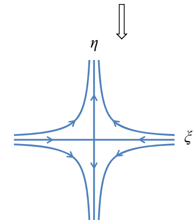
Much simpler.

Coordinate Change

Example

$$\frac{dx}{dt} = -3x + 4y, \quad \frac{dy}{dt} = -2x + 3y \;\Longrightarrow\; \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix} \;\Longrightarrow\; \frac{d\xi}{dt} = -\xi, \quad \frac{d\eta}{dt} = \eta$$





Eigenvalues

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det(A - \lambda I) = \lambda^2 - \operatorname{trace}(A)\lambda + \det(A) = \lambda^2 - \tau\lambda + \delta = (\lambda - \lambda_1)(\lambda - \lambda_2)$$
$$= \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$$

$$\lambda_1 + \lambda_2 = \operatorname{trace}(A)$$
$$\lambda_1\lambda_2 = \det(A)$$

The trace is the sum of the eigenvalues, while the determinant is the product of the eigenvalues.

We can characterize the dynamic behavior using the trace and the determinant.

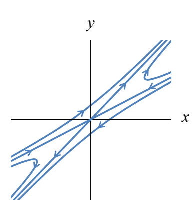
Saddles

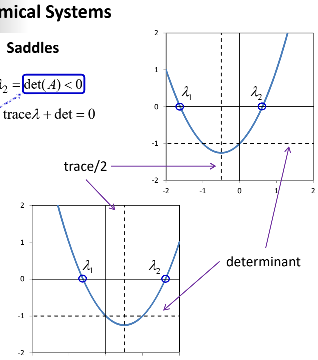
$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\lambda_1\lambda_2 = \det(A) < 0$$

$$\lambda^2 - \operatorname{trace}\lambda + \det = 0$$

One eigenvalue is positive, the other negative.





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Dynamical Systems

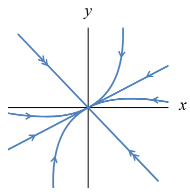
Stable Nodes

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\lambda_1\lambda_2 = \delta = \det(A) > 0$$
$$\lambda_1 + \lambda_2 = \tau = \mathrm{trace}(A) < 0$$
$$\text{discriminant} = \tau^2 - 4\delta > 0$$

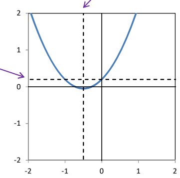
$$\lambda^2 - \tau\lambda + \delta = 0$$
$$\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}$$

Both eigenvalues are negative.



trace/2

determinant



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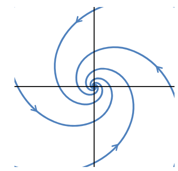
Stable Spirals

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\lambda_1\lambda_2 = \delta = \det(A) > 0$$
$$\lambda_1 + \lambda_2 = \tau = \mathrm{trace}(A) < 0$$
$$\tau^2 - 4\delta < 0$$

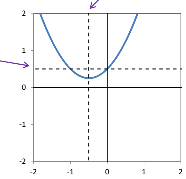
$$\lambda^2 - \tau\lambda + \delta = 0$$
$$\lambda = \frac{\tau \pm i\sqrt{4\delta - \tau^2}}{2}$$

Both eigenvalues are complex, with negative real part.



trace/2

determinant



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Dynamical Systems

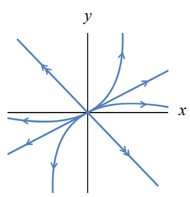
Unstable Nodes

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\lambda_1\lambda_2 = \delta = \det(A) > 0$$
$$\lambda_1 + \lambda_2 = \tau = \mathrm{trace}(A) > 0$$
$$\tau^2 - 4\delta > 0$$

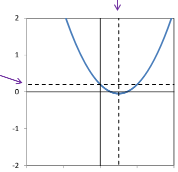
$$\lambda^2 - \tau\lambda + \delta = 0$$
$$\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}$$

Both eigenvalues are positive.



trace/2

determinant



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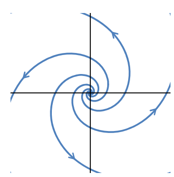
Unstable Spirals

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\lambda_1\lambda_2 = \delta = \det(A) > 0$$
$$\lambda_1 + \lambda_2 = \tau = \mathrm{trace}(A) > 0$$
$$\tau^2 - 4\delta < 0$$

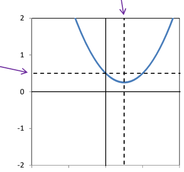
$$\lambda^2 - \tau\lambda + \delta = 0$$
$$\lambda = \frac{\tau \pm i\sqrt{4\delta - \tau^2}}{2}$$

Both eigenvalues are complex, with positive real part.



trace/2

determinant



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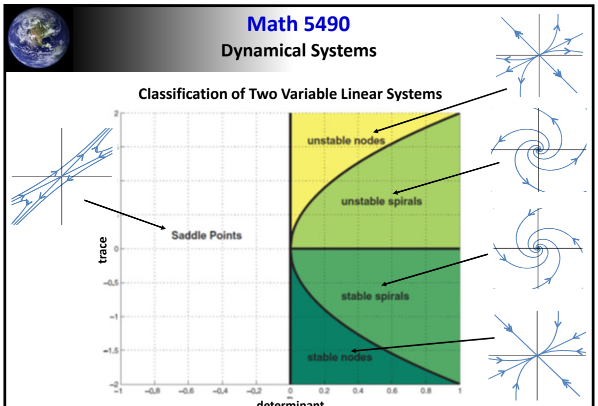
Dynamical Systems

Classification of Two Variable Linear Systems



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Dynamical Systems

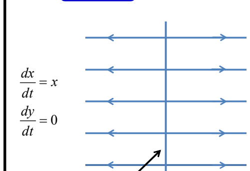
Degenerate Cases

$$\lambda_1\lambda_2 = \det(A) = 0$$
$$\lambda_1 + \lambda_2 = \mathrm{trace}(A) = \tau > 0$$

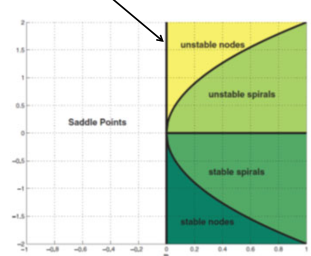
One of the eigenvalues is zero, the other positive.

$$\lambda^2 - \tau\lambda = 0$$

$$\frac{dx}{dt} = x$$
$$\frac{dy}{dt} = 0$$



rest points



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Dynamical Systems

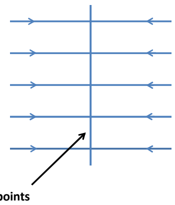
Degenerate Cases

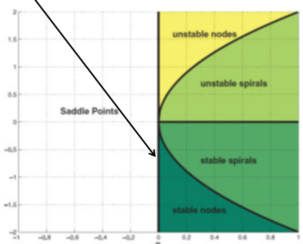
$\lambda_1\lambda_2 = \det(A) = 0$
$\lambda_1 + \lambda_2 = \text{trace}(A) = \tau < 0$

$\lambda^2 - \tau\lambda = 0$

One of the eigenvalues is zero, the other negative.

$\frac{dx}{dt} = -x$
$\frac{dy}{dt} = 0$





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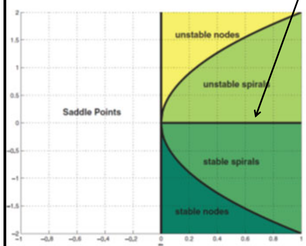
Dynamical Systems

Degenerate Cases

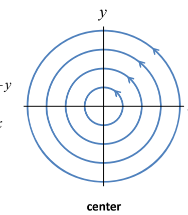
$\lambda_1 + \lambda_2 = \text{trace}(A) = 0$
$\lambda_1\lambda_2 = \det(A) = \delta > 0$

Imaginary pair of eigenvalues

$\lambda^2 + \delta = 0$



$\frac{dx}{dt} = -y$
$\frac{dy}{dt} = x$



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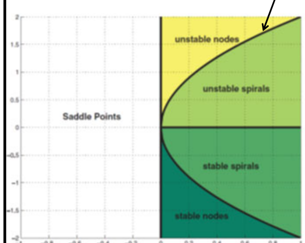
Degenerate Cases

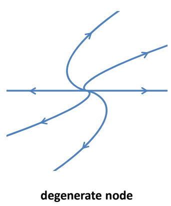
$\lambda_1\lambda_2 = \delta = \det(A) > 0$
$\lambda_1 + \lambda_2 = \tau = \text{trace}(A) > 0$
$\tau^2 - 4\delta = 0$

Positive double eigenvalue. Only one eigenvector

$\lambda^2 - \tau\lambda + \frac{\tau^2}{4} = \left(\lambda - \frac{\tau}{2}\right)^2 = 0$



$\frac{dx}{dt} = x + y$
$\frac{dy}{dt} = y$



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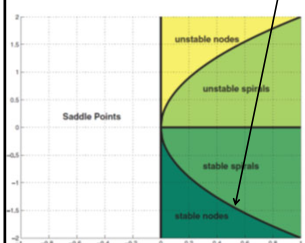
Dynamical Systems

Degenerate Cases

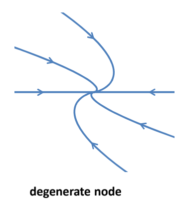
$\lambda_1\lambda_2 = \delta = \det(A) > 0$
$\lambda_1 + \lambda_2 = \tau = \text{trace}(A) < 0$
$\tau^2 - 4\delta = 0$

Negative double eigenvalue. Only one eigenvector

$\lambda^2 - \tau\lambda + \frac{\tau^2}{4} = \left(\lambda - \frac{\tau}{2}\right)^2 = 0$



$\frac{dx}{dt} = -x + y$
$\frac{dy}{dt} = -y$



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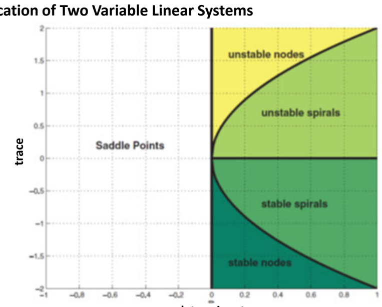
Dynamical Systems

Classification of Two Variable Linear Systems

$\frac{dx}{dt} = a_{11}x + a_{12}y$
$\frac{dy}{dt} = a_{21}x + a_{22}y$

We have now classified (given names) to the most common possible behaviors of two variable linear systems.



Kaper & Engler, 2013

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Classification of One Variable Linear Systems

Let's back up.

What is the classification of one variable linear systems?

$\frac{dx}{dt} = ax$

$a < 0$	**sink**	**stable rest point**
$a > 0$	**source**	**unstable rest point**
$a = 0$	**degenerate**	**all points resting**

degenerate

sink **source**

$a < 0$ $a = 0$ $a > 0$

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Classification of One Variable Linear Systems

Can we transform one linear equation into another with a linear change of variable?

$$\frac{dx}{dt} = ax \quad \text{to} \quad \frac{dx}{dt} = bx\ ?$$

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Classification of One Variable Linear Systems

Can we transform one linear equation into another with a linear change of variable?

$$\frac{dx}{dt} = ax \quad \text{to} \quad \frac{dx}{dt} = bx\ ?$$

Let's try.

$$x = c\xi \quad \Rightarrow \quad \frac{dx}{dt} = c\frac{d\xi}{dt} = ax = ac\xi$$

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Classification of One Variable Linear Systems

Can we transform one linear equation into another with a linear change of variable?

$$\frac{dx}{dt} = ax \quad \text{to} \quad \frac{dx}{dt} = bx\ ?$$

Let's try.

$$x = c\xi \quad \Rightarrow \quad \frac{dx}{dt} = \cancel{c}\frac{d\xi}{dt} = ax = a\cancel{c}\xi$$

Cancel c.

$$\frac{d\xi}{dt} = a\xi$$

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Classification of One Variable Linear Systems

Can we transform one linear equation into another with a linear change of variable?

$$\frac{dx}{dt} = ax \quad \text{to} \quad \frac{dx}{dt} = bx\ ?$$

Let's try.

no change!

$$x = c\xi \quad \Rightarrow \quad \frac{dx}{dt} = \cancel{c}\frac{d\xi}{dt} = ax = a\cancel{c}\xi$$

No!

Cancel c.

$$\frac{d\xi}{dt} = a\xi$$

Only if $a = b$!

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Algebraic Classification of One Variable Linear Systems

$$\frac{dx}{dt} = ax$$

They are all different.

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Algebraic Classification of Two Variable Linear Systems

$$\frac{dx}{dt} = Ax \qquad \frac{dx}{dt} = Bx$$

Assume that there are two distinct eigenvalues for both A and B.

If the eigenvalues of A are the same as those of B, then the two systems are algebraically equivalent, i.e., one can be transformed to the other by a linear change of variables.

If the eigenvalues of A are different from those of B, then the two systems are not algebraically equivalent, i.e., one cannot be transformed to the other by a linear change of variables.

The situation is more complicated for double eigenvalues.

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Dynamical Systems

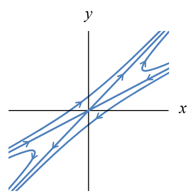
Algebraic Classification

Example

$$\frac{dx}{dt} = -3x + 4y \qquad \frac{dy}{dt} = -2x + 3y$$

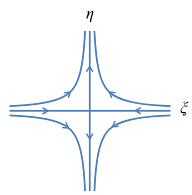
These two systems "look the same" from a linear algebra perspective, because one can be transformed to the other with a linear coordinate change.

$$\frac{d\xi}{dt} = -\xi \qquad \frac{d\eta}{dt} = \eta$$



$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

Coordinate Change



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Dynamical Systems

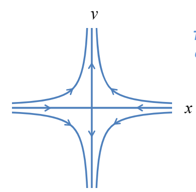
Algebraic Classification

Example

$$\frac{dx}{dt} = -2x \qquad \frac{dy}{dt} = 2y$$

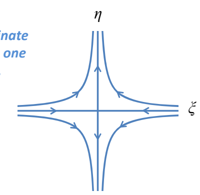
These two systems do not "look the same" from a linear algebraic perspective, because one cannot be transformed to the other with a linear coordinate change.

$$\frac{d\xi}{dt} = -\xi \qquad \frac{d\eta}{dt} = \eta$$



There is no linear coordinate change that transforms one system to the other.

Why not?



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Dynamical Systems

Algebraic Classification

Example

$$\frac{dx}{dt} = -2x \qquad \frac{dy}{dt} = 2y$$

There is no linear coordinate change that transforms one system to the other.

Why not?

$$\frac{d\xi}{dt} = -\xi \qquad \frac{d\eta}{dt} = \eta$$

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Dynamical Systems

Algebraic Classification

Example

eigenvalues -2, 2

$$\frac{dx}{dt} = -2x \qquad \frac{dy}{dt} = 2y$$

There is no linear coordinate change that transforms one system to the other.

Why not?

$$\frac{d\xi}{dt} = -\xi \qquad \frac{d\eta}{dt} = \eta$$

eigenvalues -1, 1

Let S be a nonsingular matrix, and let $\begin{bmatrix} x \\ y \end{bmatrix} = S \begin{bmatrix} \xi \\ \eta \end{bmatrix}$.

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$$

$$S\frac{d}{dt}\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \frac{d}{dt}S\begin{bmatrix} \xi \\ \eta \end{bmatrix} = AS\begin{bmatrix} \xi \\ \eta \end{bmatrix} \Longrightarrow \frac{d}{dt}\begin{bmatrix} \xi \\ \eta \end{bmatrix} = S^{-1}AS\begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

A and $S^{-1}AS$ have the same eigenvalues.

Why?

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Algebraic Classification

A and $S^{-1}AS$ have the same eigenvalues.

Why?

Let λ be an eigenvalue of A with corresponding eigenvector v, and let S be a nonsingular matrix. Then

$$\left(S^{-1}AS\right)\left(S^{-1}v\right) = S^{-1}A\left(SS^{-1}\right)v = S^{-1}Av = S^{-1}\lambda v = \lambda S^{-1}v = \lambda\left(S^{-1}v\right),$$

so λ is an eigenvalue of $S^{-1}AS$ with corresponding eigenvector $S^{-1}v$.

Therefore, if $\frac{dx}{dt} = Ax$ can be transformed to $\frac{dx}{dt} = Bx$ by a linear coordinate change, then A and B must have the same eigenvalues.

Math 5490
Dynamical Systems

Algebraic Classification

Back to Example

$$\frac{dx}{dt} = -2x \qquad \frac{dy}{dt} = 2y$$

There is no linear coordinate change that transforms one system to the other.

Why not?

$$\frac{d\xi}{dt} = -\xi \qquad \frac{d\eta}{dt} = \eta$$

$\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$ has eigenvalues -2 and 2, but $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ has eigenvalues -1 and 1.

There cannot be a linear coordinate change transforming one system to the other system.

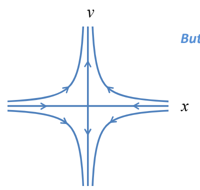
Math 5490
Dynamical Systems

Topological Classification

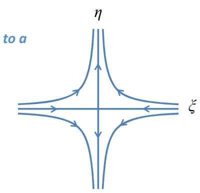
$$\frac{dx}{dt} = -2x \qquad \frac{dy}{dt} = 2y$$

These two systems do not "look the same" algebraically.

$$\frac{d\xi}{dt} = -\xi \qquad \frac{d\eta}{dt} = \eta$$



But they "look the same" to a topologist.



Math 5490
Dynamical Systems

Topological Classification

What do we mean when we say that two systems look the same to a topologist?

$$\frac{dx}{dt} = Ax \quad \Longleftrightarrow \text{ topologically conjugate? } \quad \frac{d\xi}{dt} = B\xi$$

These two systems are ***topologically conjugate*** if there is a continuous coordinate change taking one system to the other.

$$x = f(\xi) \text{ transforms } \frac{dx}{dt} = Ax \text{ to } \frac{d\xi}{dt} = B\xi.$$

Precisely what this means is beyond the scope of this course.

Math 5490
Dynamical Systems

Topological Classification

For one variable, all sources are topologically conjugate.

$$\frac{dx}{dt} = \alpha x, \quad \alpha > 0$$

$$\text{Let } x = |u|^\alpha \frac{u}{|u|} = \begin{cases} u^\alpha & u > 0 \\ -(-u)^\alpha & u < 0 \end{cases}$$

Math 5490
Dynamical Systems

Topological Classification

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$$\text{Let } x = |u|^\alpha \frac{u}{|u|} = \begin{cases} u^\alpha & u > 0 \\ -(-u)^\alpha & u < 0 \end{cases}$$

$$u > 0: \quad \frac{dx}{dt} = \alpha u^{\alpha-1}\frac{du}{dt}$$
$$= \alpha x = \alpha u^\alpha = \alpha u^{\alpha-1} u$$
$$\frac{du}{dt} = u$$

$$u < 0: \quad \frac{dx}{dt} = -\alpha(-u)^{\alpha-1}\left(-\frac{du}{dt}\right) = \alpha(-u)^{\alpha-1}\frac{du}{dt}$$
$$= \alpha x = \alpha\left(-(-u)^\alpha\right) = \alpha(-u)^{\alpha-1} u$$
$$\frac{du}{dt} = u$$

Math 5490
Dynamical Systems

Topological Classification

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$$= \alpha x = \alpha\left(-(-u)^\alpha\right) = \alpha(-u)^{\alpha-1} u$$

$$\frac{du}{dt} = u \qquad \frac{du}{dt} = u$$

Continuous, but not smooth at 0.

$$\frac{dx}{dt} = \alpha x \quad \Longrightarrow \quad x = |u|^\alpha \frac{u}{|u|} \quad \Longrightarrow \quad \frac{du}{dt} = u$$

Math 5490
Dynamical Systems

Topological Classification

For one variable, all sinks are topologically conjugate.

$$\frac{dx}{dt} = -\alpha x \quad \alpha > 0$$

$$\text{Let } x = |u|^\alpha \frac{u}{|u|} = \begin{cases} u^\alpha & u > 0 \\ -(-u)^\alpha & u < 0 \end{cases}$$

$$u > 0: \quad \frac{dx}{dt} = \alpha u^{\alpha-1}\frac{du}{dt}$$
$$= -\alpha x = -\alpha u^\alpha = -\alpha u^{\alpha-1} u$$

$$u < 0: \quad \frac{dx}{dt} = -\alpha(-u)^{\alpha-1}\left(-\frac{du}{dt}\right) = \alpha(-u)^{\alpha-1}\frac{du}{dt}$$
$$= -\alpha x = -\alpha\left(-(-u)^\alpha\right) = -\alpha(-u)^{\alpha-1} u$$

$$\frac{du}{dt} = -u \qquad \frac{du}{dt} = -u$$

Continuous, but not smooth at 0.

$$\frac{dx}{dt} = -\alpha x \quad \alpha > 0 \quad \Longrightarrow \quad x = |u|^\alpha \frac{u}{|u|} \quad \Longrightarrow \quad \frac{du}{dt} = -u$$

Math 5490
Dynamical Systems

Topological Classification for One Variable Linear Systems.

$$\frac{dx}{dt} = ax$$

There are exactly three classes.

1. **sources**: ($a > 0$) All sources are topologically conjugate.
2. **sinks**: ($a < 0$) All sinks are topologically conjugate.
3. **degenerate:** ($a > 0$) Only one case.

Math 5490
Dynamical Systems

Topological Classification of Two Variable Linear Systems

If neither eigenvalue has zero real part, then the system is called ***hyperbolic***, in which case, there are only three classes:

1. **saddles**: One positive eigenvalue and one negative. The determinant is negative.
   Any two saddles are topologically conjugate.
2. **sources**: Both eigenvalues have positive real part. The determinant is positive, and the trace is positive.
   Any two sources are topologically conjugate.
3. **sinks**: Both eigenvalues have negative real part. The determinant is positive, and the trace is negative.
   Any two sinks are topologically conjugate.

Nonhyperbolic systems are more complicated.

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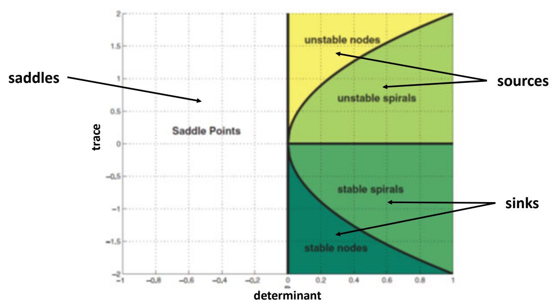
Dynamical Systems

Classification of Two Variable Linear Systems



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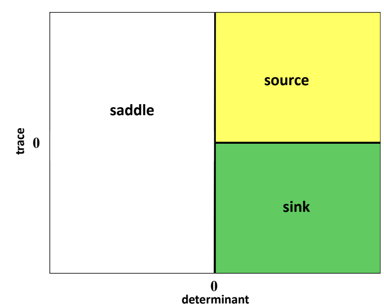
Dynamical Systems

Topological Classification

Sometimes, we only care whether a rest point is a sink, a source, or a saddle.



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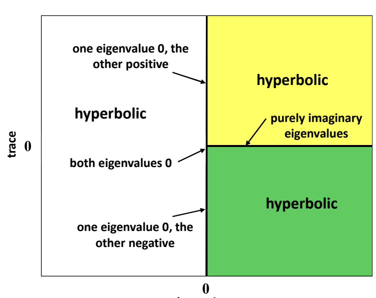
Dynamical Systems

Topological Classification: Degeneracies

The determinant is the product of the eigenvalues. When it is zero, at least one of the eigenvalues is zero.

When the trace is zero and the determinant is positive, the characteristic polynomial is
$\lambda^2 - \delta = 0,$
which implies that the eigenvalues are purely imaginary and therefore have real part zero.



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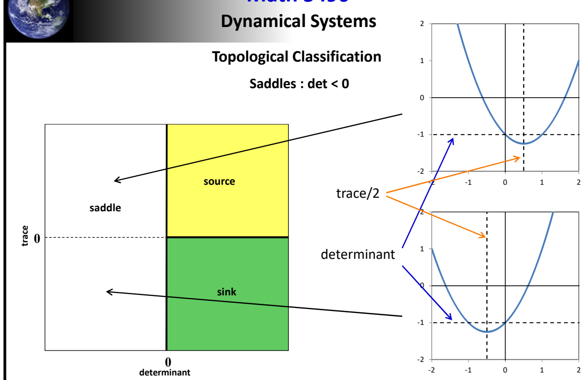
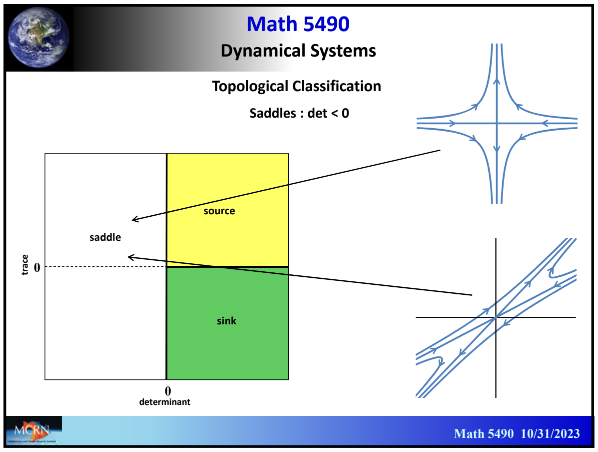
Dynamical Systems

Topological Classification

Saddles : det < 0

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Dynamical Systems
Topological Classification
Saddles : det < 0
saddle
source
sink
trace
0
0
determinant
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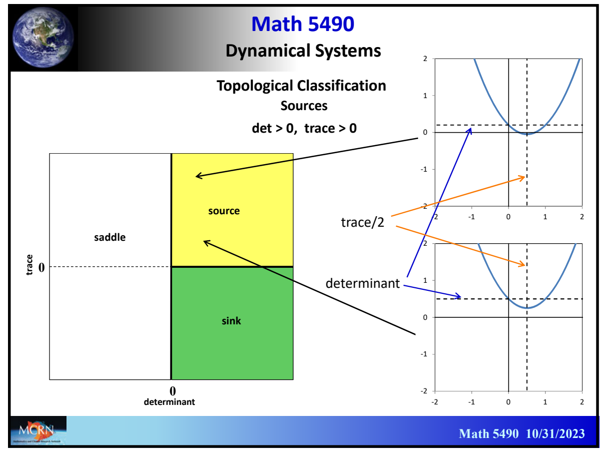
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Topological Classification
Sources
det > 0, trace > 0
saddle
source
sink
trace
determinant
trace/2
determinant
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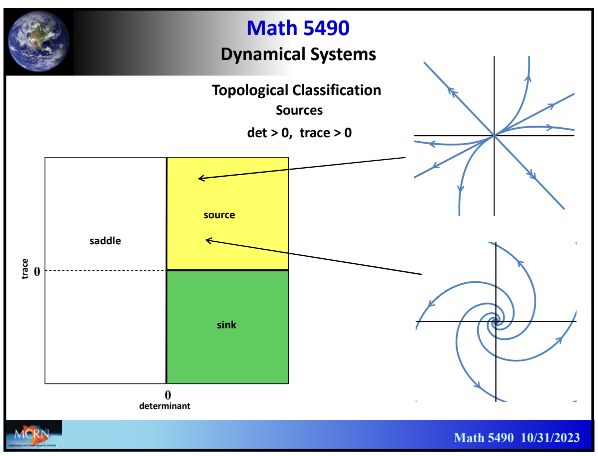
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Topological Classification
Sources
det > 0, trace > 0
saddle
source
sink
trace
determinant
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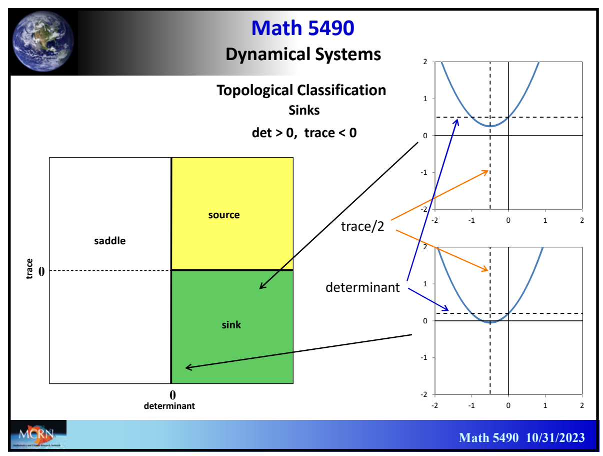
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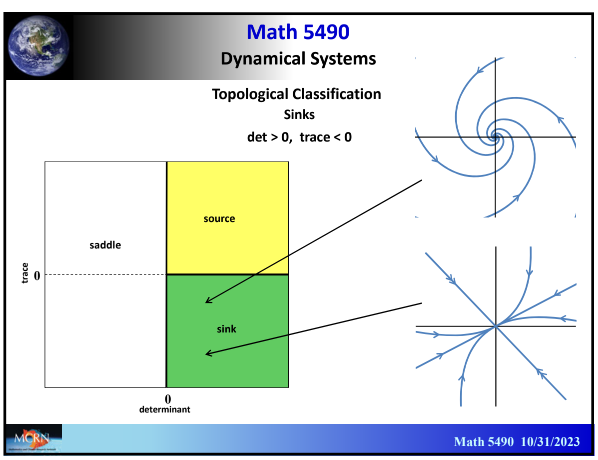
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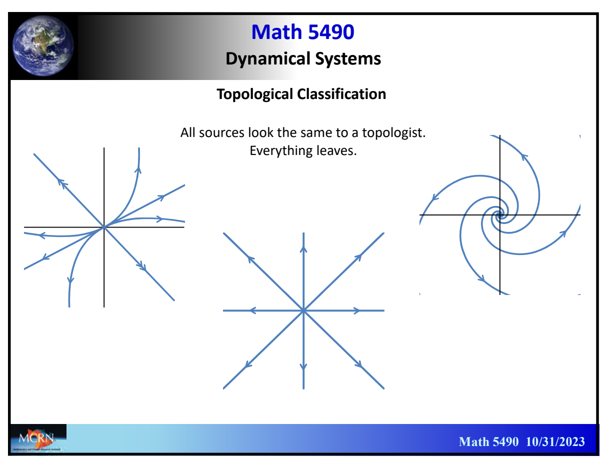
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Dynamical Systems
Topological Classification
All sources look the same to a topologist.
Everything leaves.
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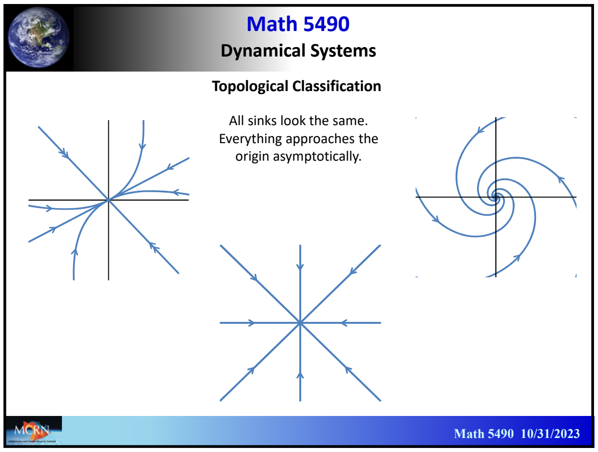
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Dynamical Systems
Topological Classification
All sinks look the same.
Everything approaches the origin asymptotically.
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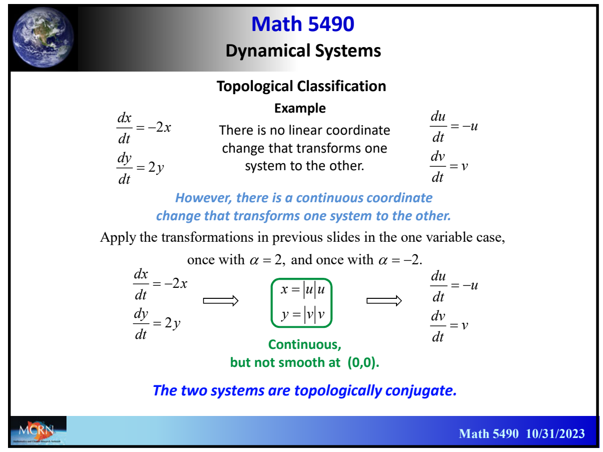
Math 5490
Dynamical Systems
Topological Classification
Example
$\frac{dx}{dt} = -2x$, $\frac{dy}{dt} = 2y$
There is no linear coordinate change that transforms one system to the other.
$\frac{du}{dt} = -u$, $\frac{dv}{dt} = v$
However, there is a continuous coordinate change that transforms one system to the other.
Apply the transformations in previous slides in the one variable case, once with $\alpha = 2$, and once with $\alpha = -2$.
$\frac{dx}{dt} = -2x$, $\frac{dy}{dt} = 2y$ ⟹ $x = |u|u$, $y = |v|v$ ⟹ $\frac{du}{dt} = -u$, $\frac{dv}{dt} = v$
Continuous, but not smooth at (0,0).
The two systems are topologically conjugate.
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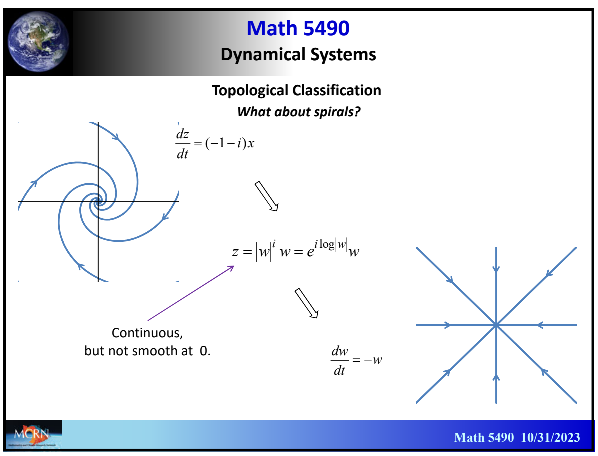
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Dynamical Systems
Topological Classification
What about spirals?
$\frac{dz}{dt} = (-1-i)x$
$z = |w|^i w = e^{i\log|w|}w$
Continuous, but not smooth at 0.
$\frac{dw}{dt} = -w$
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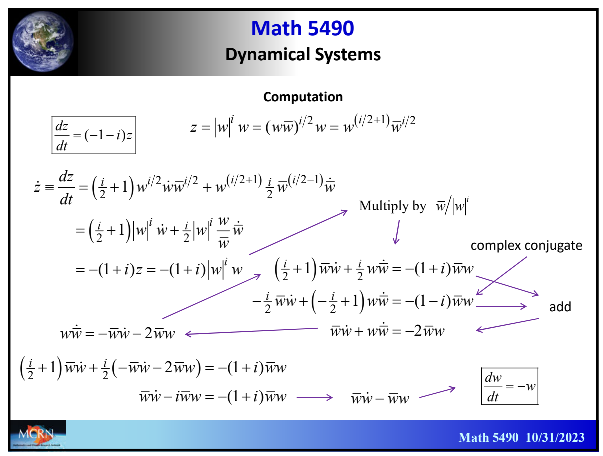
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Dynamical Systems
Computation
$\frac{dz}{dt} = (-1-i)z$
$z = |w|^i w = (w\overline{w})^{i/2} w = w^{(i/2+1)}\overline{w}^{i/2}$
$\dot{z} \equiv \frac{dz}{dt} = \left(\frac{i}{2}+1\right) w^{i/2}\dot{w}\overline{w}^{i/2} + w^{(i/2+1)}\,\frac{i}{2}\,\overline{w}^{(i/2-1)}\dot{\overline{w}}$
$= \left(\frac{i}{2}+1\right)|w|^i\,\dot{w} + \frac{i}{2}|w|^i\,\frac{w}{\overline{w}}\dot{\overline{w}}$
$= -(1+i)z = -(1+i)|w|^i\,w$
Multiply by $\overline{w}/|w|^i$
$\left(\frac{i}{2}+1\right)\overline{w}\dot{w} + \frac{i}{2}w\dot{\overline{w}} = -(1+i)\overline{w}w$
complex conjugate
$-\frac{i}{2}\overline{w}\dot{w} + \left(-\frac{i}{2}+1\right)w\dot{\overline{w}} = -(1-i)\overline{w}w$
add
$\overline{w}\dot{w} + w\dot{\overline{w}} = -2\overline{w}w$
$w\dot{\overline{w}} = -\overline{w}\dot{w} - 2\overline{w}w$
$\left(\frac{i}{2}+1\right)\overline{w}\dot{w} + \frac{i}{2}\left(-\overline{w}\dot{w} - 2\overline{w}w\right) = -(1+i)\overline{w}w$
$\overline{w}\dot{w} - i\overline{w}w = -(1+i)\overline{w}w$ ⟶ $\overline{w}\dot{w} - \overline{w}w$ ⟶ $\frac{dw}{dt} = -w$
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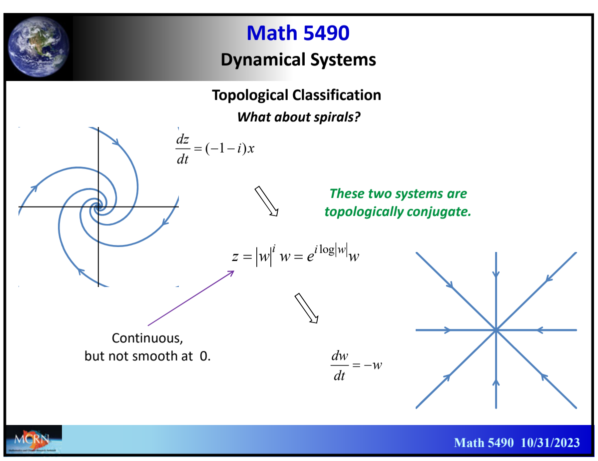
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Dynamical Systems
Topological Classification
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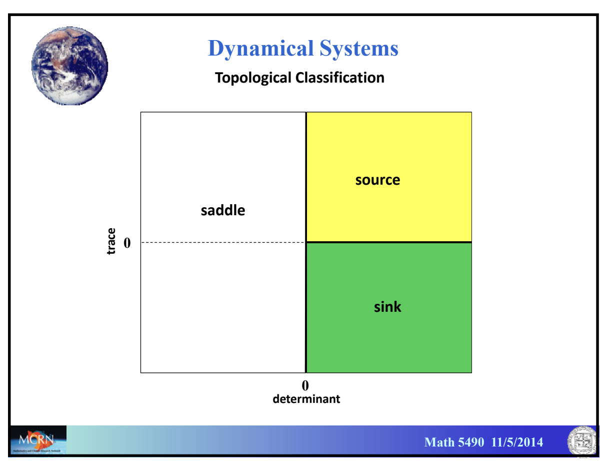
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Topological Classification
saddle
source
sink
trace
0
0
determinant
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