## Math 5490
### Topics in Applied Mathematics
# Introduction to the Mathematics of Climate

**Fall 2023**
**1:25 - 3:20 Tuesdays and Thursdays**
**Amundson Hall 162**

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## Math 5490
### Dynamical Systems

The dependence of the solution on ***initial conditions*** is just as important as its dependence on ***time***.

$x \in \mathbb{R}^n, \quad \xi \in \mathbb{R}^n, \quad f : \mathbb{R}^n \to \mathbb{R}^n$

**intital value problem** $\dot{x} = f(x)$, $x(0) = \xi$

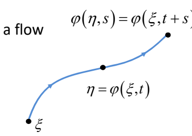
The initial value problem generates a flow

$\varphi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$

with properties

**intital condition** $\varphi(\xi, 0) = \xi$

**"group property"** $\varphi(\varphi(\xi,t),s) = \varphi(\xi, t+s)$




If we start the system at state $\xi$ and follow the solution for time $t$, then restart the system at the new state and follow the solution for time $s$, we end up at the same state as starting at $\xi$ and following for time $t+s$.

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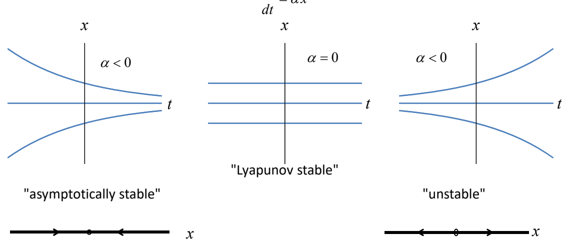
## Math 5490
### Dynamical Systems

**Example**

$$\frac{dx}{dt} = \alpha x$$

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## Math 5490
### Dynamical Systems

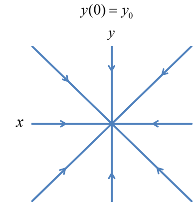
**"Phase Plane"**

**Example**

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$\dot{x} = -x, \quad \dot{y} = -y, \quad x(0) = x_0, \quad y(0) = y_0$$

$$\varphi\left(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, t\right) = \begin{bmatrix} x_0 e^{-t} \\ y_0 e^{-t} \end{bmatrix}$$




$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

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## Math 5490
### Dynamical Systems

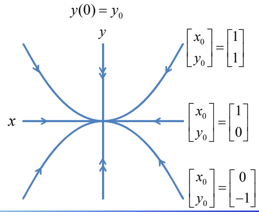
**"Phase Plane"**

**Example**

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$\dot{x} = -x, \quad \dot{y} = -2y, \quad x(0) = x_0, \quad y(0) = y_0$$

$$\varphi\left(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, t\right) = \begin{bmatrix} x_0 e^{-t} \\ y_0 e^{-2t} \end{bmatrix}$$

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$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

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## Math 5490
### Dynamical Systems

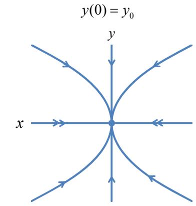
**"Phase Plane"**

**Example**

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$\dot{x} = -2x, \quad \dot{y} = -y, \quad x(0) = x_0, \quad y(0) = y_0$$

$$\varphi\left(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, t\right) = \begin{bmatrix} x_0 e^{-2t} \\ y_0 e^{-t} \end{bmatrix}$$




$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

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## Math 5490
### Dynamical Systems

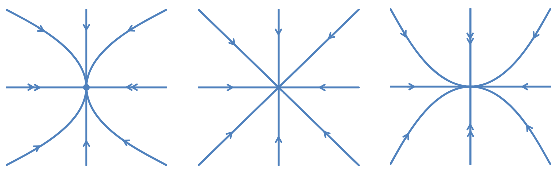
**"Phase Plane"**

**Example: "stable nodes"**

$$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}a&0\\0&b\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}\qquad \begin{bmatrix}x\\y\end{bmatrix}(0)=\begin{bmatrix}x_0\\y_0\end{bmatrix}$$

$$\dot x = ax,\quad \dot y = by,\quad x(0)=x_0,\quad y(0)=y_0$$

$$\varphi\left(\begin{bmatrix}x_0\\y_0\end{bmatrix},t\right)=\begin{bmatrix}x_0e^{at}\\y_0e^{bt}\end{bmatrix}$$



$a<b<0$ $\qquad$ $a=b<0$ $\qquad$ $b<a<0$



## Math 5490
### Dynamical Systems

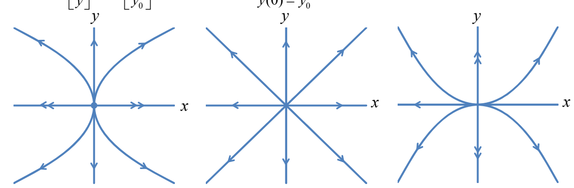
**"Phase Plane"**

**Example: "unstable nodes"**

$$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}a&0\\0&b\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}\qquad \begin{bmatrix}x\\y\end{bmatrix}(0)=\begin{bmatrix}x_0\\y_0\end{bmatrix}$$

$$\dot x = ax,\quad \dot y = by,\quad x(0)=x_0,\quad y(0)=y_0$$

$$\varphi\left(\begin{bmatrix}x_0\\y_0\end{bmatrix},t\right)=\begin{bmatrix}x_0e^{at}\\y_0e^{bt}\end{bmatrix}$$




$a>b>0$ $\qquad$ $a=b>0$ $\qquad$ $0<a<b$



## Math 5490
### Dynamical Systems

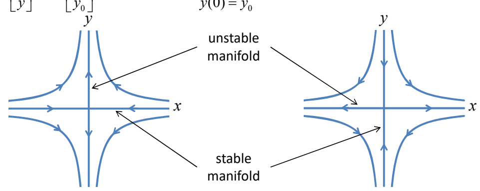
**"Phase Plane"**

**Example: "saddles"**

$$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}a&0\\0&b\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}\qquad \begin{bmatrix}x\\y\end{bmatrix}(0)=\begin{bmatrix}x_0\\y_0\end{bmatrix}$$

$$\dot x = ax,\quad \dot y = by,\quad x(0)=x_0,\quad y(0)=y_0$$

$$\varphi\left(\begin{bmatrix}x_0\\y_0\end{bmatrix},t\right)=\begin{bmatrix}x_0e^{at}\\y_0e^{bt}\end{bmatrix}$$




$a<0<b$ $\qquad$ $b<0<a$



## Math 5490
### Dynamical Systems

**Matrix Notation**

$$\begin{aligned}\frac{dx}{dt}&=a_{11}x+a_{12}y\\ \frac{dy}{dt}&=a_{21}x+a_{22}y\end{aligned}\quad\Longrightarrow\quad \mathbf{x}=\begin{bmatrix}x\\y\end{bmatrix}\quad A=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\quad\Longrightarrow\quad \frac{d\mathbf{x}}{dt}=A\mathbf{x}$$

Amazingly, the solution is

$$\mathbf{x}(t)=e^{At}\mathbf{x}_0$$

**Example**

$$\begin{aligned}\frac{dx}{dt}&=\alpha x\\ \frac{dy}{dt}&=\beta y\end{aligned}\quad\Longrightarrow\quad \mathbf{x}=\begin{bmatrix}x\\y\end{bmatrix}\quad A=\begin{bmatrix}\alpha&0\\0&\beta\end{bmatrix}\quad\Longrightarrow\quad \frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\alpha&0\\0&\beta\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

$$\begin{bmatrix}x\\y\end{bmatrix}(t)=\begin{bmatrix}e^{\alpha t}&0\\0&e^{\beta t}\end{bmatrix}\begin{bmatrix}x_0\\y_0\end{bmatrix}=\begin{bmatrix}e^{\alpha t}x_0\\e^{\beta t}y_0\end{bmatrix}$$



## Math 5490
### Dynamical Systems

**Eigenvalues**

$$Av=\lambda v\qquad (v\neq 0)$$

The number $\lambda$ is called an ***eigenvalue*** and the vector $v$ is called an ***eigenvector***.

**Example**

$$A=\begin{bmatrix}\alpha&0\\0&\beta\end{bmatrix}$$

$$A\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}\alpha&0\\0&\beta\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}\alpha\\0\end{bmatrix}=\alpha\begin{bmatrix}1\\0\end{bmatrix}$$

$$A\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}\alpha&0\\0&\beta\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\\beta\end{bmatrix}=\beta\begin{bmatrix}0\\1\end{bmatrix}$$

$\alpha$ and $\beta$ are eigenvalues with corresponding eigenvectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\0\end{bmatrix}$.



## Math 5490
### Dynamical Systems

**Eigenvalues**

**Example**

$$A=\begin{bmatrix}1&2\\1&0\end{bmatrix}$$

$$A\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}1&2\\1&0\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}4\\2\end{bmatrix}=2\begin{bmatrix}2\\1\end{bmatrix}$$

$$A\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}1&2\\1&0\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix}=(-1)\begin{bmatrix}-1\\1\end{bmatrix}$$

$2$ and $-1$ are eigenvalues with corresponding eigenvectors $\begin{bmatrix}2\\1\end{bmatrix}$ and $\begin{bmatrix}-1\\1\end{bmatrix}$.

## Math 5490
### Dynamical Systems

**Eigenvalues**

*What are they good for?*

To find solutions of

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

If $\lambda$ is an eigenvalue of $A$ with corresponding eigenvector $v$,

then $x(t) = e^{\lambda t} v$ is a solution.

Proof:

$$\frac{dx}{dt} = \lambda e^{\lambda t} v = e^{\lambda t} \lambda v = e^{\lambda t} A v = A\left(e^{\lambda t} v\right) = Ax$$



## Math 5490
### Dynamical Systems

**Eigenvalues**

**Example**

$$\frac{dx}{dt} = \alpha x \qquad \frac{dy}{dt} = \beta y \qquad \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

eigenvalues: $\alpha$ and $\beta$

eigenvectors: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Solutions: $\begin{bmatrix} x \\ y \end{bmatrix}(t) = e^{\alpha t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{\alpha t} \\ 0 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}(t) = e^{\beta t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ e^{\beta t} \end{bmatrix}$



## Math 5490
### Dynamical Systems

**Eigenvalues**

**Example**

$$\frac{dx}{dt} = x + 2y \qquad \frac{dy}{dt} = x \qquad \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

eigenvalues: $2$ and $-1$

eigenvectors: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Solutions: $\begin{bmatrix} x \\ y \end{bmatrix}(t) = e^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ e^{2t} \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}(t) = e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix}$



## Math 5490
### Dynamical Systems

**Eigenvalues**

***More good news:***

If $x = \varphi_1(t)$ and $x = \varphi_2(t)$ are solutions of $\frac{dx}{dt} = Ax$,

then $x(t) = c_1\varphi_1(t) + c_2\varphi_2(t)$ is also a solution for arbitrary constants $c_1$ and $c_2$.

Proof:

$$\begin{aligned} \frac{dx}{dt} &= \frac{d}{dt}\left(c_1\varphi_1(t) + c_2\varphi_2(t)\right) \\ &= c_1\varphi_1{}'(t) + c_2\varphi_2{}'(t) = c_1 A\varphi_1(t) + c_2 A\varphi_2(t) \\ &= A\left(c_1\varphi_1(t) + c_2\varphi_2(t)\right) \\ &= Ax \end{aligned}$$



## Math 5490
### Dynamical Systems

**Eigenvalues**

**Example**

$$\frac{dx}{dt} = \alpha x \qquad \frac{dy}{dt} = \beta y \qquad \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Solutions: $\begin{bmatrix} x \\ y \end{bmatrix}(t) = \begin{bmatrix} e^{\alpha t} \\ 0 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}(t) = \begin{bmatrix} 0 \\ e^{\beta t} \end{bmatrix}$

General solution: $\begin{bmatrix} x \\ y \end{bmatrix}(t) = c_1\begin{bmatrix} e^{\alpha t} \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ e^{\beta t} \end{bmatrix} = \begin{bmatrix} c_1 e^{\alpha t} \\ c_2 e^{\beta t} \end{bmatrix}$

$$\boxed{\begin{aligned} x(t) &= c_1 e^{\alpha t} \\ y(t) &= c_2 e^{\beta t} \end{aligned}}$$



## Math 5490
### Dynamical Systems

**Eigenvalues**

**Example**

$$\frac{dx}{dt} = x + 2y \qquad \frac{dy}{dt} = x \qquad \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Solutions: $\begin{bmatrix} x \\ y \end{bmatrix}(t) = \begin{bmatrix} 2e^{2t} \\ e^{2t} \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}(t) = \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix}$

General solution: $\begin{bmatrix} x \\ y \end{bmatrix}(t) = c_1\begin{bmatrix} 2e^{2t} \\ e^{2t} \end{bmatrix} + c_2\begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} 2c_1 e^{2t} - c_2 e^{-t} \\ c_1 e^{2t} + c_2 e^{-t} \end{bmatrix}$

$$\boxed{\begin{aligned} x(t) &= 2c_1 e^{2t} - c_2 e^{-t} \\ y(t) &= c_1 e^{2t} + c_2 e^{-t} \end{aligned}}$$

## Math 5490
### Dynamical Systems

**Eigenvalues**

$$Av = \lambda v$$

***How do we find the eigenvalues?***

$$Av = \lambda v \quad \Rightarrow \quad Av - \lambda v = 0 \quad \Rightarrow \quad (A - \lambda I)v = 0$$

For $(A-\lambda I)v = 0$ to have a nontrival solution $v \neq 0$, we must have

$$\boxed{\det(A - \lambda I) = 0.}$$

***Characteristic Polynomial***

The roots of the characteristic polynomial are the eigenvalues of $A$.



## Math 5490
### Dynamical Systems

**Eigenvalues**

The roots of the characteristic polynomial are the eigenvalues.

**Example**

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} \alpha - \lambda & 0 \\ 0 & \beta - \lambda \end{bmatrix} = (\alpha - \lambda)(\beta - \lambda)$$

The eigenvalues are the roots of $(\lambda - \alpha)(\lambda - \beta) = 0$, namely,
$\alpha$ and $\beta$ .



## Math 5490
### Dynamical Systems

**Eigenvalues**

**Example**

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 2 \\ 1 & 0-\lambda \end{bmatrix} = (1-\lambda)(0-\lambda) - 2$$
$$= \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$$

The eigenvalues are the roots of $(\lambda - 2)(\lambda + 1) = 0$, namely,
$2$ and $-1$ .



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### Dynamical Systems

**Eigenvalues**
**In general**

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21}$$
$$= \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}$$
$$= \lambda^2 - \operatorname{trace}(A)\lambda + \det(A)$$
$$= \lambda^2 - \tau\lambda + \delta$$

The eigenvalues are $\lambda = \dfrac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}$

real if $\tau^2 - 4\delta \geq 0$
complex if $\tau^2 - 4\delta < 0$



## Math 5490
### Dynamical Systems

**Eigenvalues**

**Example**

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

$$\tau = \operatorname{trace}(A) = 1 + 0 = 1$$
$$\delta = \det(A) = 1 \cdot 0 - 2 \cdot 1 = -2$$

$$\det(A - \lambda I) = \lambda^2 - \tau\lambda + \delta = \lambda^2 - \lambda - 2$$

The eigenvalues are $\lambda = \dfrac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2} = \dfrac{1 \pm \sqrt{1^2 - 4(-2)}}{2} = \dfrac{1 \pm \sqrt{9}}{2} = \dfrac{1 \pm 3}{2}$

$$\boxed{2 \text{ and } -1}$$



## Math 5490
### Dynamical Systems

**Eigenvalues**

$$Av = \lambda v$$

***How do we find the eigenvectors?***

$$Av = \lambda v \quad \Rightarrow \quad Av - \lambda v = 0 \quad \Rightarrow \quad (A - \lambda I)v = 0$$

If we have an eigenvalue $\lambda$ we can find a corresponding eigenvector by solving
$(A - \lambda I)v = 0$
for a nontrival solution $v$.

## Math 5490
### Dynamical Systems

**Eigenvalues**

**Example** $A=\begin{bmatrix}1 & 2\\ 1 & 0\end{bmatrix}$ $\lambda_1=2,\quad \lambda_2=-1$ ← eigenvalues

$$A-\lambda_1 I=\begin{bmatrix}1-\lambda_1 & 2\\ 1 & 0-\lambda_1\end{bmatrix}=\begin{bmatrix}1-2 & 2\\ 1 & 0-2\end{bmatrix}=\begin{bmatrix}-1 & 2\\ 1 & -2\end{bmatrix}$$

$$(A-\lambda_1 I)v_1=\begin{bmatrix}-1 & 2\\ 1 & -2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=0 \qquad \lambda_1=2 \quad v_1=\begin{bmatrix}2\\ 1\end{bmatrix}$$

corresponding eigenvectors

$$A-\lambda_2 I=\begin{bmatrix}1-\lambda_2 & 2\\ 1 & 0-\lambda_2\end{bmatrix}=\begin{bmatrix}1-(-1) & 2\\ 1 & 0-(-1)\end{bmatrix}=\begin{bmatrix}2 & 2\\ 1 & 1\end{bmatrix}$$

$$(A-\lambda_2 I)v_2=\begin{bmatrix}2 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=0 \qquad \lambda_2=-1 \quad v_2=\begin{bmatrix}-1\\ 1\end{bmatrix}$$

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## Math 5490
### Dynamical Systems

**Eigenvalues**

**Example**

$$\frac{dx}{dt}=-2y \qquad \frac{dy}{dt}=x-3y \qquad \frac{d}{dt}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0 & -2\\ 1 & -3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=A\begin{bmatrix}x\\ y\end{bmatrix} \qquad A=\begin{bmatrix}0 & -2\\ 1 & -3\end{bmatrix}$$

$$\tau=\operatorname{trace}(A)=0-3=-3$$
$$\delta=\det(A)=0\cdot(-3)-(-2)\cdot 1=2$$

$$\det(A-\lambda I)=\lambda^2-\tau\lambda+\delta=\lambda^2+3\lambda+2=(\lambda+1)(\lambda+2)$$

The eigenvalues are $\boxed{-1 \text{ and } -2}$

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## Math 5490
### Dynamical Systems

**Eigenvalues**

$A=\begin{bmatrix}0 & -2\\ 1 & -3\end{bmatrix}$ $\lambda_1=-1,\quad \lambda_2=-2$ ← eigenvalues

$$A-\lambda_1 I=\begin{bmatrix}0-\lambda_1 & -2\\ 1 & -3-\lambda_1\end{bmatrix}=\begin{bmatrix}0-(-1) & -2\\ 1 & -3-(-1)\end{bmatrix}=\begin{bmatrix}1 & -2\\ 1 & -2\end{bmatrix}$$

corresponding eigenvectors

$$(A-\lambda_1 I)v_1=\begin{bmatrix}1 & -2\\ 1 & -2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=0 \qquad v_1=\begin{bmatrix}2\\ 1\end{bmatrix}$$

$$A-\lambda_2 I=\begin{bmatrix}0-\lambda_2 & -2\\ 1 & -3-\lambda_2\end{bmatrix}=\begin{bmatrix}0-(-2) & -2\\ 1 & -3-(-2)\end{bmatrix}=\begin{bmatrix}2 & -2\\ 1 & -1\end{bmatrix}$$

$$(A-\lambda_2 I)v_2=\begin{bmatrix}2 & -2\\ 1 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=0 \qquad v_2=\begin{bmatrix}1\\ 1\end{bmatrix}$$

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## Math 5490
### Dynamical Systems

**Eigenvalues**

**Example**

$$\frac{dx}{dt}=-2y \qquad \frac{dy}{dt}=x-3y \qquad \frac{d}{dt}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0 & -2\\ 1 & -3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=A\begin{bmatrix}x\\ y\end{bmatrix} \qquad A=\begin{bmatrix}0 & -2\\ 1 & -3\end{bmatrix}$$

eigenvalues: $-1 \quad -2$

eigenvectors: $\begin{bmatrix}2\\ 1\end{bmatrix} \quad \begin{bmatrix}1\\ 1\end{bmatrix}$

Solutions: $\begin{bmatrix}x\\ y\end{bmatrix}(t)=e^{-t}\begin{bmatrix}2\\ 1\end{bmatrix}$ and $\begin{bmatrix}x\\ y\end{bmatrix}(t)=e^{-2t}\begin{bmatrix}1\\ 1\end{bmatrix}$

General solution: $\begin{bmatrix}x\\ y\end{bmatrix}(t)=c_1\begin{bmatrix}2e^{-t}\\ e^{-t}\end{bmatrix}+c_2\begin{bmatrix}e^{-2t}\\ e^{-2t}\end{bmatrix}=\begin{bmatrix}2c_1e^{-t}+c_2e^{-2t}\\ c_1e^{-t}+c_2e^{-2t}\end{bmatrix}$

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## Math 5490
### Dynamical Systems

**Eigenvalues**

**Example**

$$\frac{dx}{dt}=-y \qquad \frac{dy}{dt}=x \qquad \frac{d}{dt}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=A\begin{bmatrix}x\\ y\end{bmatrix} \qquad A=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

$$\tau=\operatorname{trace}(A)=0+0=0$$
$$\delta=\det(A)=0\cdot 0-(-1)\cdot 1=1$$

$$\det(A-\lambda I)=\lambda^2-\tau\lambda+\delta=\lambda^2+1=(\lambda-i)(\lambda+i)$$

The eigenvalues are $\boxed{i \text{ and } -i}$

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## Math 5490
### Dynamical Systems

**Eigenvalues**

$A=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ $\lambda_1=i,\quad \lambda_2=-i$ ← eigenvalues

corresponding eigenvectors

$$A-\lambda_1 I=\begin{bmatrix}0-\lambda_1 & -1\\ 1 & 0-\lambda_1\end{bmatrix}=\begin{bmatrix}0-i & -1\\ 1 & 0-i\end{bmatrix}=\begin{bmatrix}-i & -1\\ 1 & -i\end{bmatrix}$$

$$(A-\lambda_1 I)v_1=\begin{bmatrix}-i & -1\\ 1 & -i\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=0 \qquad v_1=\begin{bmatrix}i\\ 1\end{bmatrix}$$

$$A-\lambda_2 I=\begin{bmatrix}0-\lambda_2 & -1\\ 1 & 0-\lambda_2\end{bmatrix}=\begin{bmatrix}0-(-i) & -1\\ 1 & 0-(-i)\end{bmatrix}=\begin{bmatrix}i & -1\\ 1 & i\end{bmatrix}$$

$$(A-\lambda_2 I)v_2=\begin{bmatrix}i & -1\\ 1 & i\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=0 \qquad v_2=\begin{bmatrix}-i\\ 1\end{bmatrix}$$

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## Math 5490
### Dynamical Systems

#### Eigenvalues

**Example**

$$\frac{dx}{dt} = -y \qquad \frac{dy}{dt} = x \qquad \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} \qquad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

eigenvalues: $i \quad -i$

eigenvectors: $\begin{bmatrix} i \\ 1 \end{bmatrix} \quad \begin{bmatrix} -i \\ 1 \end{bmatrix}$

**Reality Check:**

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ i \end{bmatrix} = i\begin{bmatrix} i \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -i \end{bmatrix} = -i\begin{bmatrix} -i \\ 1 \end{bmatrix}$$



## Math 5490
### Dynamical Systems

#### Eigenvalues

**Example**

$$\frac{dx}{dt} = -y \qquad \frac{dy}{dt} = x \qquad \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} \qquad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

eigenvalues: $i \quad -i$

eigenvectors: $\begin{bmatrix} i \\ 1 \end{bmatrix} \quad \begin{bmatrix} -i \\ 1 \end{bmatrix}$

Solutions: $\begin{bmatrix} x \\ y \end{bmatrix}(t) = e^{it}\begin{bmatrix} i \\ 1 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}(t) = e^{-it}\begin{bmatrix} -i \\ 1 \end{bmatrix}$

General solution: $\begin{bmatrix} x \\ y \end{bmatrix}(t) = c_1\begin{bmatrix} ie^{it} \\ e^{it} \end{bmatrix} + c_2\begin{bmatrix} -ie^{-it} \\ e^{-it} \end{bmatrix} = \begin{bmatrix} ic_1e^{it} - ic_2e^{-it} \\ c_1e^{it} + c_2e^{-it} \end{bmatrix}$



## Math 5490
### Dynamical Systems

#### Eigenvalues

**Example**

$$\frac{dx}{dt} = -y \qquad \frac{dy}{dt} = x \qquad \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} \qquad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

General solution: $\begin{bmatrix} x \\ y \end{bmatrix}(t) = c_1\begin{bmatrix} ie^{it} \\ e^{it} \end{bmatrix} + c_2\begin{bmatrix} -ie^{-it} \\ e^{-it} \end{bmatrix} = \begin{bmatrix} ic_1e^{it} - ic_2e^{-it} \\ c_1e^{it} + c_2e^{-it} \end{bmatrix}$

Let $c_1 = \frac{1}{2i}$, $c_2 = -\frac{1}{2i}$. Then $\begin{bmatrix} x \\ y \end{bmatrix}(t) = \begin{bmatrix} \frac{1}{2}(e^{it} + e^{-it}) \\ \frac{1}{2i}(e^{it} - e^{-it}) \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$

Let $c_1 = c_2 = \frac{1}{2}$. Then $\begin{bmatrix} x \\ y \end{bmatrix}(t) = \begin{bmatrix} \frac{i}{2}(e^{it} - e^{-it}) \\ \frac{1}{2}(e^{it} + e^{-it}) \end{bmatrix} = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$



## Math 5490
### Dynamical Systems

#### Eigenvalues

**Example**

$$\frac{dx}{dt} = -y \qquad \frac{dy}{dt} = x \qquad \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} \qquad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

General solution: $\begin{bmatrix} x \\ y \end{bmatrix}(t) = c_1\begin{bmatrix} ie^{it} \\ e^{it} \end{bmatrix} + c_2\begin{bmatrix} -ie^{-it} \\ e^{-it} \end{bmatrix} = \begin{bmatrix} ic_1e^{it} - ic_2e^{-it} \\ c_1e^{it} + c_2e^{-it} \end{bmatrix}$

Or $\begin{bmatrix} x \\ y \end{bmatrix}(t) = a\begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + b\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$



## Math 5490
### Dynamical Systems

#### Eigenvalues

**Alternate Approach**

$$\frac{dx}{dt} = -y \qquad \frac{dy}{dt} = x$$

Let $z = x + iy$. Then $\frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt} = -y + ix = iz$

$\frac{dz}{dt} = iz$ Solution: $z(t) = z_0e^{it} = r_0e^{i\theta_0}e^{it} = r_0e^{i(\theta_0 + t)}$

Let $z = re^{i\theta}$. Then $\frac{dz}{dt} = \frac{dr}{dt}e^{i\theta} + rie^{i\theta}\frac{d\theta}{dt} = iz = ire^{i\theta}$

$$\frac{dr}{dt} + ri\frac{d\theta}{dt} = 0 + ir$$

$\frac{dr}{dt} = 0$, $\frac{d\theta}{dt} = 1$ Solution: $r(t) = r_0$, $\theta(t) = \theta_0 + t$



## Math 5490
### Dynamical Systems

#### Summary So Far

$$\frac{dx}{dt} = a_{11}x + a_{12}y \qquad \frac{dy}{dt} = a_{21}x + a_{22}y \quad \Longrightarrow \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \Longrightarrow \quad \frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

Eigenvalues and eigenvectors

$$Av = \lambda v \qquad (v \neq 0)$$

If $v$ and $u$ are linearly independent eigenvectors with corresponding eigenvalues $\lambda$ and $\mu$, then the general solution is

$$\mathbf{x}(t) = c_1e^{\lambda t}v + c_2e^{\mu t}u$$

where $c_1$ and $c_2$ are arbitrary constants.

***Linear independence***: one is not a multiple of the other.

## Math 5490
### Dynamical Systems

**Coordinate Change**

$$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix}=A\begin{bmatrix}x\\y\end{bmatrix}$$

Suppose that $\boldsymbol{v}=\begin{bmatrix}v_1\\v_2\end{bmatrix}$ and $\boldsymbol{u}=\begin{bmatrix}u_1\\u_2\end{bmatrix}$ are linearly independent eigenvectors of $A$ with corresponding eigenvalues $\lambda$ and $\mu$. Introduce new variables $\xi$ and $\eta$:

$$\begin{bmatrix}x\\y\end{bmatrix}=\xi\boldsymbol{v}+\eta\boldsymbol{u},\quad \text{i.e.}\quad \begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}v_1\xi+u_1\eta\\v_2\xi+u_2\eta\end{bmatrix}=\begin{bmatrix}v_1 & u_1\\v_2 & u_2\end{bmatrix}\begin{bmatrix}\xi\\\eta\end{bmatrix}=S\begin{bmatrix}\xi\\\eta\end{bmatrix}$$

where $S=\begin{bmatrix}v_1 & u_1\\v_2 & u_2\end{bmatrix}=\begin{bmatrix}\boldsymbol{v}\,|\,\boldsymbol{u}\end{bmatrix}$.

Then $S\frac{d}{dt}\begin{bmatrix}\xi\\\eta\end{bmatrix}=\frac{d}{dt}S\begin{bmatrix}\xi\\\eta\end{bmatrix}=\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix}=A\begin{bmatrix}x\\y\end{bmatrix}=AS\begin{bmatrix}\xi\\\eta\end{bmatrix}$

$$\frac{d}{dt}\begin{bmatrix}\xi\\\eta\end{bmatrix}=S^{-1}AS\begin{bmatrix}\xi\\\eta\end{bmatrix}$$

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## Math 5490
### Dynamical Systems

**Coordinate Change**

$$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix}=A\begin{bmatrix}x\\y\end{bmatrix}\implies\begin{bmatrix}x\\y\end{bmatrix}=S\begin{bmatrix}\xi\\\eta\end{bmatrix}\implies\frac{d}{dt}\begin{bmatrix}\xi\\\eta\end{bmatrix}=S^{-1}AS\begin{bmatrix}\xi\\\eta\end{bmatrix}$$

$$S=\begin{bmatrix}\boldsymbol{v}\,|\,\boldsymbol{u}\end{bmatrix}\qquad A\boldsymbol{v}=\lambda\boldsymbol{v}\qquad A\boldsymbol{u}=\mu\boldsymbol{u}$$

$$AS=A\begin{bmatrix}\boldsymbol{v}\,|\,\boldsymbol{u}\end{bmatrix}=\begin{bmatrix}A\boldsymbol{v}\,|\,A\boldsymbol{u}\end{bmatrix}=\begin{bmatrix}\lambda\boldsymbol{v}\,|\,\mu\boldsymbol{u}\end{bmatrix}=\begin{bmatrix}\boldsymbol{v}\,|\,\boldsymbol{u}\end{bmatrix}\begin{bmatrix}\lambda & 0\\0 & \mu\end{bmatrix}=S\Lambda$$

$$A\begin{bmatrix}\boldsymbol{v}\,|\,\boldsymbol{u}\end{bmatrix}=\begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}v_1 & u_1\\v_2 & u_2\end{bmatrix}=\begin{bmatrix}a_{11}v_1+a_{12}v_2 & a_{11}u_1+a_{12}u_2\\a_{21}v_1+a_{22}v_2 & a_{21}u_1+a_{22}u_2\end{bmatrix}=\begin{bmatrix}A\boldsymbol{v}\,|\,A\boldsymbol{u}\end{bmatrix}$$

$$\begin{bmatrix}\lambda\boldsymbol{v}\,|\,\mu\boldsymbol{u}\end{bmatrix}=\begin{bmatrix}\lambda v_1 & \mu u_1\\\lambda v_2 & \mu u_2\end{bmatrix}=\begin{bmatrix}v_1 & u_1\\v_2 & u_2\end{bmatrix}\begin{bmatrix}\lambda & 0\\0 & \mu\end{bmatrix}=\begin{bmatrix}\boldsymbol{v}\,|\,\boldsymbol{u}\end{bmatrix}\begin{bmatrix}\lambda & 0\\0 & \mu\end{bmatrix}$$

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## Math 5490
### Dynamical Systems

**Coordinate Change**

$$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix}=A\begin{bmatrix}x\\y\end{bmatrix}\implies\begin{bmatrix}x\\y\end{bmatrix}=S\begin{bmatrix}\xi\\\eta\end{bmatrix}\implies\frac{d}{dt}\begin{bmatrix}\xi\\\eta\end{bmatrix}=S^{-1}AS\begin{bmatrix}\xi\\\eta\end{bmatrix}$$

$$S=\begin{bmatrix}\boldsymbol{v}\,|\,\boldsymbol{u}\end{bmatrix}\qquad A\boldsymbol{v}=\lambda\boldsymbol{v}\qquad A\boldsymbol{u}=\mu\boldsymbol{u}$$

$$AS=A\begin{bmatrix}\boldsymbol{v}\,|\,\boldsymbol{u}\end{bmatrix}=\begin{bmatrix}A\boldsymbol{v}\,|\,A\boldsymbol{u}\end{bmatrix}=\begin{bmatrix}\lambda\boldsymbol{v}\,|\,\mu\boldsymbol{u}\end{bmatrix}=\begin{bmatrix}\boldsymbol{v}\,|\,\boldsymbol{u}\end{bmatrix}\begin{bmatrix}\lambda & 0\\0 & \mu\end{bmatrix}=S\Lambda$$

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## Math 5490
### Dynamical Systems

**Coordinate Change**

$$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix}=A\begin{bmatrix}x\\y\end{bmatrix}\implies\begin{bmatrix}x\\y\end{bmatrix}=S\begin{bmatrix}\xi\\\eta\end{bmatrix}\implies\frac{d}{dt}\begin{bmatrix}\xi\\\eta\end{bmatrix}=S^{-1}AS\begin{bmatrix}\xi\\\eta\end{bmatrix}$$

$$S=\begin{bmatrix}\boldsymbol{v}\,|\,\boldsymbol{u}\end{bmatrix}\qquad A\boldsymbol{v}=\lambda\boldsymbol{v}\qquad A\boldsymbol{u}=\mu\boldsymbol{u}$$

$$AS=A\begin{bmatrix}\boldsymbol{v}\,|\,\boldsymbol{u}\end{bmatrix}=\begin{bmatrix}A\boldsymbol{v}\,|\,A\boldsymbol{u}\end{bmatrix}=\begin{bmatrix}\lambda\boldsymbol{v}\,|\,\mu\boldsymbol{u}\end{bmatrix}=\begin{bmatrix}\boldsymbol{v}\,|\,\boldsymbol{u}\end{bmatrix}\begin{bmatrix}\lambda & 0\\0 & \mu\end{bmatrix}=S\Lambda$$

$$\Lambda=S^{-1}AS$$

$$\frac{d}{dt}\begin{bmatrix}\xi\\\eta\end{bmatrix}=S^{-1}AS\begin{bmatrix}\xi\\\eta\end{bmatrix}=\Lambda\begin{bmatrix}\xi\\\eta\end{bmatrix}$$

$$\begin{aligned}\frac{dx}{dt}&=a_{11}x+a_{12}y\\\frac{dy}{dt}&=a_{21}x+a_{22}y\end{aligned}\quad\implies\quad\begin{aligned}\frac{d\xi}{dt}&=\lambda\xi\\\frac{d\eta}{dt}&=\mu\eta\end{aligned}$$

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## Math 5490
### Dynamical Systems

**Coordinate Change**

**Example**

$$\begin{aligned}\frac{dx}{dt}&=x+2y\\\frac{dy}{dt}&=x\end{aligned}\qquad A=\begin{bmatrix}1 & 2\\1 & 0\end{bmatrix}$$

eigenvalues: $2$ and $-1$

eigenvectors: $\begin{bmatrix}2\\1\end{bmatrix}$ and $\begin{bmatrix}-1\\1\end{bmatrix}$.

$$S=\begin{bmatrix}2 & -1\\1 & 1\end{bmatrix}\qquad \Lambda=\begin{bmatrix}2 & 0\\0 & -1\end{bmatrix}$$

$$\begin{bmatrix}x\\y\end{bmatrix}=S\begin{bmatrix}\xi\\\eta\end{bmatrix}=\begin{bmatrix}2 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}\xi\\\eta\end{bmatrix}=\begin{bmatrix}2\xi-\eta\\\xi+\eta\end{bmatrix}\qquad\begin{aligned}x&=2\xi-\eta\\y&=\xi+\eta\end{aligned}$$

$$\begin{aligned}\frac{dx}{dt}&=x+2y\\\frac{dy}{dt}&=x\end{aligned}\quad\implies\quad\begin{aligned}x&=2\xi-\eta\\y&=\xi+\eta\end{aligned}\quad\implies\quad\begin{aligned}\frac{d\xi}{dt}&=2\xi\\\frac{d\eta}{dt}&=-\eta\end{aligned}$$

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## Math 5490
### Dynamical Systems

**Coordinate Change**

**Example**

$$\begin{aligned}\frac{dx}{dt}&=-y\\\frac{dy}{dt}&=x\end{aligned}\qquad A=\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$$

eigenvalues: $i$ $\;-i$

eigenvectors: $\begin{bmatrix}i\\1\end{bmatrix}$ $\begin{bmatrix}-i\\1\end{bmatrix}$

$$S=\begin{bmatrix}i & -i\\1 & 1\end{bmatrix}\qquad \Lambda=\begin{bmatrix}i & 0\\0 & -i\end{bmatrix}$$

$$\begin{bmatrix}x\\y\end{bmatrix}=S\begin{bmatrix}z\\w\end{bmatrix}=\begin{bmatrix}i & -i\\1 & 1\end{bmatrix}\begin{bmatrix}z\\w\end{bmatrix}=\begin{bmatrix}iz-iw\\z+w\end{bmatrix}\qquad\begin{aligned}x&=iz-iw\\y&=z+w\end{aligned}$$

$$\begin{aligned}\frac{dx}{dt}&=-y\\\frac{dy}{dt}&=x\end{aligned}\quad\implies\quad\begin{aligned}x&=iz-iw\\y&=z+w\end{aligned}\quad\implies\quad\begin{aligned}\frac{dz}{dt}&=iz\\\frac{dw}{dt}&=-iw\end{aligned}$$

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## Math 5490
### Dynamical Systems

**Coordinate Change**

**Example**

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = x \quad \Longrightarrow \quad x = iz - iw, \quad y = z + w \quad \Longrightarrow \quad \frac{dz}{dt} = iz, \quad \frac{dw}{dt} = -iw$$

Note that one of these equations is redundant.

$$2z = y - ix, \quad 2w = y + ix \quad \Rightarrow \quad w = \bar{z}$$

| Cartesian | complex | polar |
| --- | --- | --- |
| $\frac{dx}{dt} = -y$, $\frac{dy}{dt} = x$ | $\frac{dz}{dt} = iz$ | $\frac{dr}{dt} = 0$, $\frac{d\theta}{dt} = 1$ |



## Math 5490
### Dynamical Systems

**Coordinate Change**

**Example**

$$\frac{dx}{dt} = ax - \omega y, \quad \frac{dy}{dt} = \omega x + ay \qquad \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & -\omega \\ \omega & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} \qquad A = \begin{bmatrix} a & -\omega \\ \omega & a \end{bmatrix}$$

$$\tau = \operatorname{trace}(A) = a + a = 2a$$
$$\delta = \det(A) = a^2 + \omega^2$$

$$\det(A - \lambda I) = \lambda^2 - \tau\lambda + \delta = \lambda^2 - 2a + a^2 + \omega^2$$

The eigenvalues are $\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2} = \frac{2a \pm \sqrt{4a^2 - 4a^2 - 4\omega^2}}{2} = a \pm i\omega$

$\boxed{a + i\omega \text{ and } a - i\omega}$



## Math 5490
### Dynamical Systems

**Coordinate Change**

$$\frac{dx}{dt} = ax - \omega y, \quad \frac{dy}{dt} = \omega x + ay$$

Let $z = x + iy$.

Then $\frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt} = ax - \omega y + i\omega x + iay = (a + i\omega)(x + iy)$

$\boxed{\frac{dz}{dt} = (a + i\omega)z}$ Solution: $z(t) = z_0 e^{(a+i\omega)t} = r_0 e^{i\theta_0} e^{(a+i\omega)t} = r_0 e^{at} e^{i(\omega t + \theta_0)}$

Let $z = re^{i\theta}$.

Then $\frac{dz}{dt} = \frac{dr}{dt}e^{i\theta} + rie^{i\theta}\frac{d\theta}{dt} = (a + i\omega)z = (a + i\omega)re^{i\theta}$

$\frac{dr}{dt} + ri\frac{d\theta}{dt} = (a + i\omega)r$

$\boxed{\frac{dr}{dt} = ar, \quad \frac{d\theta}{dt} = \omega}$ Solution: $r(t) = r_0 e^{at}$, $\theta(t) = \theta_0 + \omega t$



## Math 5490
### Dynamical Systems

**Complex Eigenvalues**

If $A$ is a matrix with real elements and
if $\lambda$ is an eigenvalue of $A$ with corrresponding eigenvector $v$,
then $\bar{\lambda}$ is an eigenvalue of $A$ with corresponding eigenvector $\bar{v}$.

$$Av = \lambda v \quad \Rightarrow \quad \overline{A}\bar{v} = \bar{\lambda}\bar{v} \quad \Rightarrow \quad A\bar{v} = \bar{\lambda}\bar{v}$$

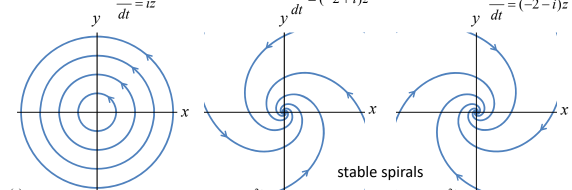


## Math 5490
### Dynamical Systems

**Stable Spirals**

$$\frac{dx}{dt} = Ax$$

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\frac{dz}{dt} = iz$ | $A = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$, $\frac{dz}{dt} = (-2 + i)z$ | $A = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix}$, $\frac{dz}{dt} = (-2 - i)z$




$r(t) = r_0$, $\theta(t) = \theta_0 + t$ | $r(t) = r_0 e^{-2t}$, $\theta(t) = \theta_0 + t$ | $r(t) = r_0 e^{-2t}$, $\theta(t) = \theta_0 - t$

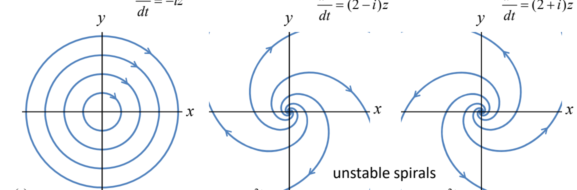


## Math 5490
### Dynamical Systems

**Unstable Spirals**

$$\frac{dx}{dt} = Ax$$

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\frac{dz}{dt} = -iz$ | $A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$, $\frac{dz}{dt} = (2 - i)z$ | $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$, $\frac{dz}{dt} = (2 + i)z$




$r(t) = r_0$, $\theta(t) = \theta_0 - t$ | $r(t) = r_0 e^{2t}$, $\theta(t) = \theta_0 - t$ | $r(t) = r_0 e^{2t}$, $\theta(t) = \theta_0 + t$

## Math 5490
### Dynamical Systems

**Coordinate Change**

**Example**

$$\frac{dx}{dt} = -4x + 2y \qquad \frac{dy}{dt} = x - 5y$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} \qquad A = \begin{bmatrix} -4 & 2 \\ 1 & -5 \end{bmatrix}$$

$$\tau = \text{trace}(A) = -4 - 5 = -9$$

$$\delta = \det(A) = (-4)\cdot(-5) - 2\cdot 1 = 18$$

$$\det(A - \lambda I) = \lambda^2 - \tau\lambda + \delta = \lambda^2 + 9\lambda + 18 = (\lambda+3)(\lambda+6)$$

The eigenvalues are $\lambda = -3$ and $\lambda = -6$.

$$A + 3I = \begin{bmatrix} -4+3 & 2 \\ 1 & -5+3 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \qquad A + 6I = \begin{bmatrix} -4+6 & 2 \\ 1 & -5+6 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$

$$\lambda = -3 \quad v = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \lambda = -6 \quad v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} -3 & 0 \\ 0 & -6 \end{bmatrix} \quad S\Lambda = AS \quad \Lambda = S^{-1}AS$$

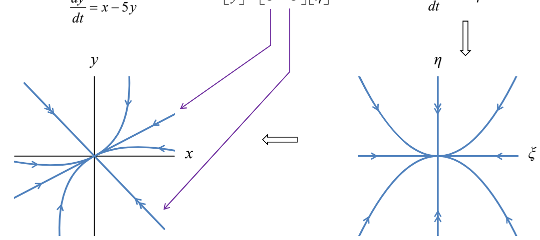


## Math 5490
### Dynamical Systems

**Coordinate Change**

**Example**

$$\frac{dx}{dt} = -4x + 2y, \quad \frac{dy}{dt} = x - 5y \;\Longrightarrow\; \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix} \;\Longrightarrow\; \frac{d\xi}{dt} = -3\xi, \quad \frac{d\eta}{dt} = -6\eta$$






## Math 5490
### Dynamical Systems

**Coordinate Change**

**Example**

$$\frac{dx}{dt} = -3x + 4y \qquad \frac{dy}{dt} = -2x + 3y$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} \qquad A = \begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix}$$

$$\tau = \text{trace}(A) = -3 + 3 = 0$$

$$\delta = \det(A) = (-3)\cdot 3 - 4\cdot(-2) = -1$$

$$\det(A - \lambda I) = \lambda^2 - \tau\lambda + \delta = \lambda^2 - 1 = (\lambda+1)(\lambda-1)$$

The eigenvalues are $\lambda = -1$ and $\lambda = 1$.

$$A + I = \begin{bmatrix} -3+1 & 4 \\ -2 & 3+1 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -2 & 4 \end{bmatrix} \qquad A - I = \begin{bmatrix} -3-1 & 4 \\ -2 & 3-1 \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ -2 & 2 \end{bmatrix}$$

$$\lambda = -1 \quad v = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \lambda = 1 \quad v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad S\Lambda = AS \quad \Lambda = S^{-1}AS$$



## Math 5490
### Dynamical Systems

**Coordinate Change**

**Example**

$$\frac{dx}{dt} = -3x + 4y, \quad \frac{dy}{dt} = -2x + 3y \;\Longrightarrow\; \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix} \;\Longrightarrow\; \frac{d\xi}{dt} = -\xi, \quad \frac{d\eta}{dt} = \eta$$