Math 5490
Topics in Applied Mathematics
Introduction to the Mathematics of Climate

Fall 2023
1:25 - 3:20 Tuesdays and Thursdays
Amundson Hall 162

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course website
www-users.cse.umn.edu/~mcgehee/teaching/Math5490/

Math 5490
Ocean Circulation

Welander's Model

Reference

Pierre Welander, A simple heat-salt oscillator, *Dynamics of Atmospheres and Oceans* **6** (1982) 233-242.

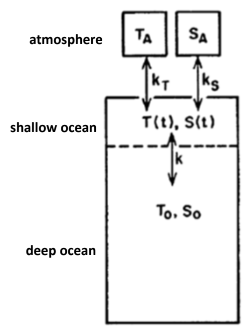
Math 5490
Ocean Circulation

The "deep ocean" is assumed to have a constant temperature T_0 and a constant salinity S_0.

The interaction between the shallow ocean and the atmosphere is modeled as a dynamic transfer of relaxation to equilibrium:

$$\frac{dT}{dt} = k_T(T_A - T),$$
$$\frac{dS}{dt} = k_S(S_A - S),$$

where k_T and k_S are constants.



Math 5490
Ocean Circulation

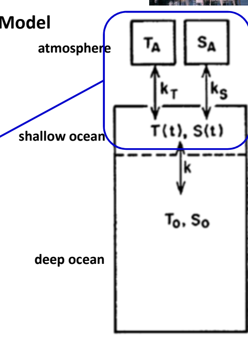
Welander's Model

The interaction between the shallow ocean and the atmosphere is modeled as a dynamic transfer of relaxation to equilibrium:

$$\frac{dT}{dt} = k_T(T_A - T),$$
$$\frac{dS}{dt} = k_S(S_A - S),$$

where k_T and k_S are positive constants. This system has a stable equilibrium point at

$$(T,S) = (T_A, S_A).$$



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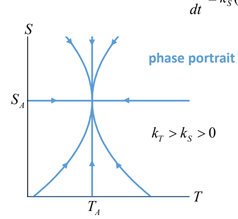
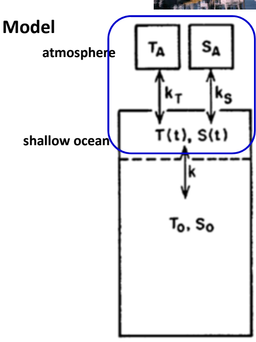
Ocean Circulation

Welander's Model

$$\frac{dT}{dt} = k_T(T_A - T),$$
$$\frac{dS}{dt} = k_S(S_A - S),$$





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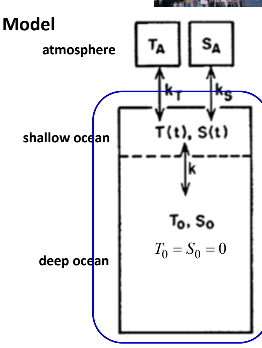
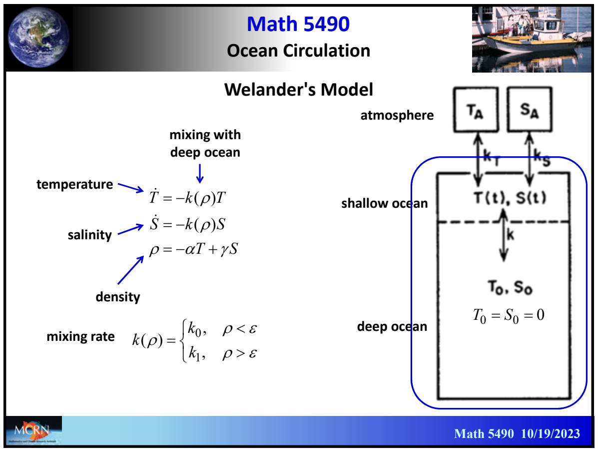
Ocean Circulation

Welander's Model

The interaction between the shallow ocean and the deep ocean is both simpler and more complex. The coordinates are chosen so that its temperature and salinity at both zero. The complication is that the mixing is determined by the density, which is a function of the temperature and salinity.

Welander simplified this complexity down to an assumption that the mixing is either zero (when the density of the shallow ocean is large) and one (when the density is low).

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Ocean Circulation
Welander's Model
mixing with deep ocean
temperature
salinity
density
$\dot{T} = -k(\rho)T$
$\dot{S} = -k(\rho)S$
$\rho = -\alpha T + \gamma S$
mixing rate $k(\rho) = \begin{cases} k_0, & \rho < \varepsilon \\ k_1, & \rho > \varepsilon \end{cases}$
atmosphere
T_A S_A
k_T k_S
shallow ocean
T(t), S(t)
k
T_0, S_0
$T_0 = S_0 = 0$
deep ocean
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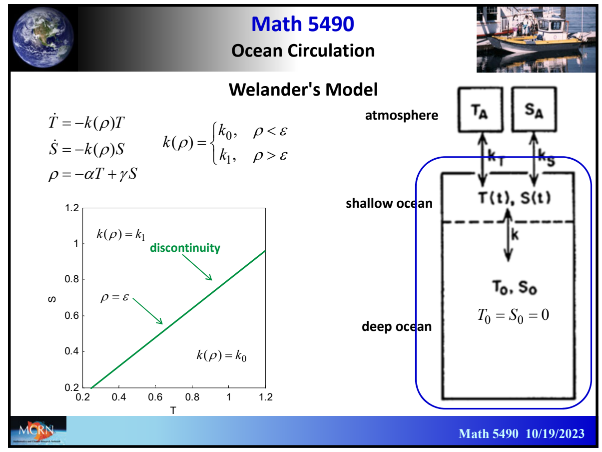
Math 5490
Ocean Circulation
Welander's Model
$\dot{T} = -k(\rho)T$
$\dot{S} = -k(\rho)S$
$\rho = -\alpha T + \gamma S$
$k(\rho) = \begin{cases} k_0, & \rho < \varepsilon \\ k_1, & \rho > \varepsilon \end{cases}$
$k(\rho) = k_1$
discontinuity
$\rho = \varepsilon$
$k(\rho) = k_0$
S
T
atmosphere
shallow ocean
deep ocean
T(t), S(t)
$T_0 = S_0 = 0$
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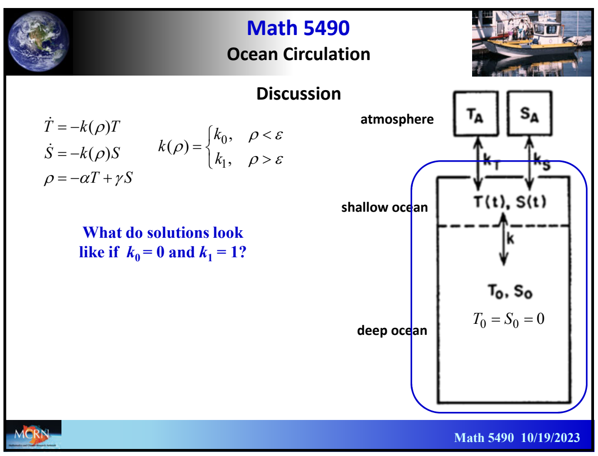
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Ocean Circulation
Discussion
$\dot{T} = -k(\rho)T$
$\dot{S} = -k(\rho)S$
$\rho = -\alpha T + \gamma S$
$k(\rho) = \begin{cases} k_0, & \rho < \varepsilon \\ k_1, & \rho > \varepsilon \end{cases}$
What do solutions look like if $k_0 = 0$ and $k_1 = 1$?
atmosphere
shallow ocean
deep ocean
T(t), S(t)
$T_0 = S_0 = 0$
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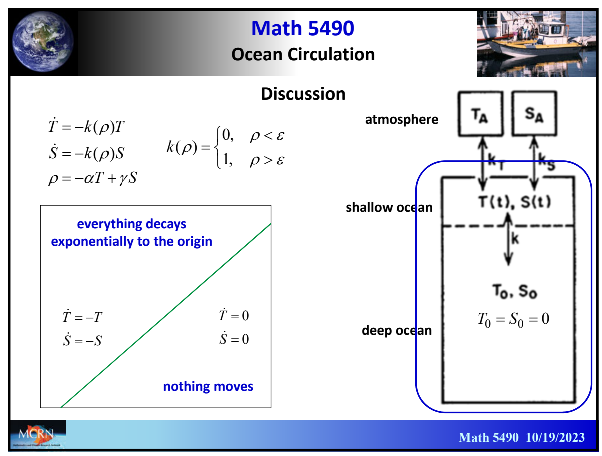
Math 5490
Ocean Circulation
Discussion
$\dot{T} = -k(\rho)T$
$\dot{S} = -k(\rho)S$
$\rho = -\alpha T + \gamma S$
$k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$
everything decays exponentially to the origin
$\dot{T} = -T$
$\dot{S} = -S$
$\dot{T} = 0$
$\dot{S} = 0$
nothing moves
atmosphere
shallow ocean
deep ocean
T(t), S(t)
$T_0 = S_0 = 0$
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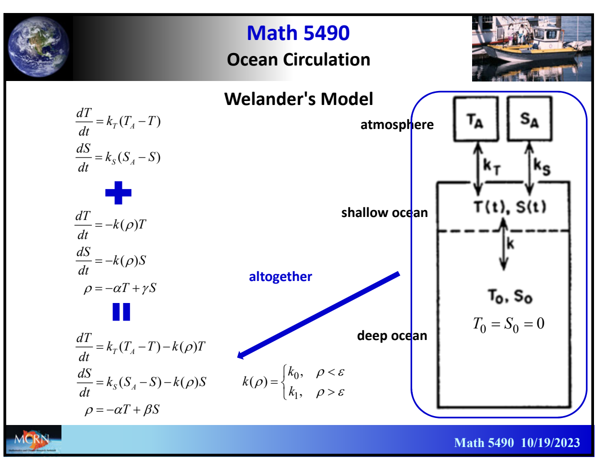
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Ocean Circulation
Welander's Model
$\frac{dT}{dt} = k_T(T_A - T)$
$\frac{dS}{dt} = k_S(S_A - S)$
+
$\frac{dT}{dt} = -k(\rho)T$
$\frac{dS}{dt} = -k(\rho)S$
$\rho = -\alpha T + \gamma S$
=
$\frac{dT}{dt} = k_T(T_A - T) - k(\rho)T$
$\frac{dS}{dt} = k_S(S_A - S) - k(\rho)S$
$\rho = -\alpha T + \beta S$
altogether
$k(\rho) = \begin{cases} k_0, & \rho < \varepsilon \\ k_1, & \rho > \varepsilon \end{cases}$
atmosphere
shallow ocean
deep ocean
T(t), S(t)
$T_0 = S_0 = 0$
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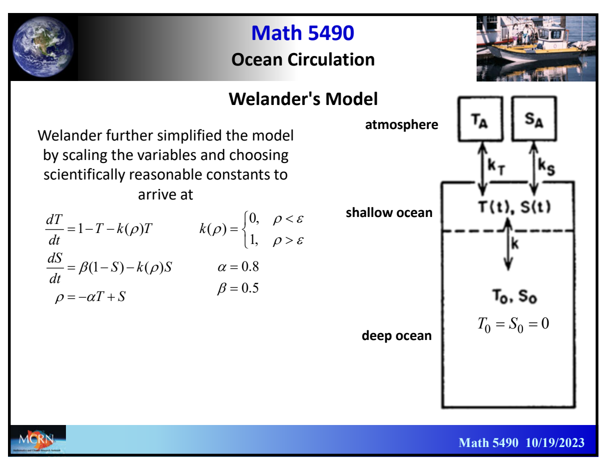
Math 5490
Ocean Circulation
Welander's Model
Welander further simplified the model by scaling the variables and choosing scientifically reasonable constants to arrive at
$\frac{dT}{dt} = 1 - T - k(\rho)T$
$\frac{dS}{dt} = \beta(1 - S) - k(\rho)S$
$\rho = -\alpha T + S$
$k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$
$\alpha = 0.8$
$\beta = 0.5$
atmosphere
shallow ocean
deep ocean
T(t), S(t)
$T_0 = S_0 = 0$
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Math 5490
Ocean Circulation

Welander's Model

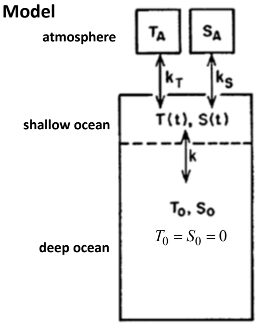
Welander further simplified the model by scaling the variables and choosing scientifically reasonable constants to arrive at

$$\frac{dT}{dt} = 1 - T - k(\rho)T \qquad k(\rho) = \begin{cases} 0, & \rho < \boxed{\varepsilon} \\ 1, & \rho > \boxed{\varepsilon} \end{cases}$$

$$\frac{dS}{dt} = \boxed{\beta}(1-S) - k(\rho)S \qquad \alpha = 0.8$$

$$\rho = -\boxed{\alpha}T + S \qquad \beta = 0.5$$

Three essential parameters: α, β, and ε



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Ocean Circulation

Scaling Idea

$$\frac{dx}{dt} = -ax + bx^2 \qquad \text{two parameters}$$

equilibria: $-ax + bx^2 = 0 \qquad x = 0, \text{ and } x = a/b$

$$\longrightarrow \underset{0}{\bullet} \longleftarrow \underset{a/b}{\bullet} \longrightarrow x$$

scale: $x = c\xi \qquad \dfrac{dx}{dt} = c\dfrac{d\xi}{dt} = -ax + bx^2 = -ac\xi + bc^2\xi^2$

$$\frac{d\xi}{dt} = -a\xi + bc\xi^2$$

choose scaling constant: $c = a/b \qquad \dfrac{d\xi}{dt} = -a\left(\xi - \xi^2\right)$ only one essential parameter

$$\longrightarrow \underset{0}{\bullet} \longleftarrow \underset{1}{\bullet} \longrightarrow \xi$$

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Ocean Circulation

Welander's Model

$$\frac{dT}{dt} = 1 - T - k(\rho)T \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$$

$$\frac{dS}{dt} = \beta(1-S) - k(\rho)S \qquad \alpha = 0.8$$

$$\rho = -\alpha T + S \qquad \beta = 0.5$$

equilibria

$$1 - T - k(\rho)T = 0$$

$$\beta(1-S) - k(\rho)S = 0$$

two cases

$\rho < \varepsilon$, so $k(\rho) = 0$

$$1 - T = 0$$

$$\beta(1-S) = 0$$

$$\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\rho > \varepsilon$, so $k(\rho) = 1$

$$1 - 2T = 0$$

$$\beta - (\beta+1)S = 0$$

$$\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} 1/2 \\ \beta/(\beta+1) \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix}$$

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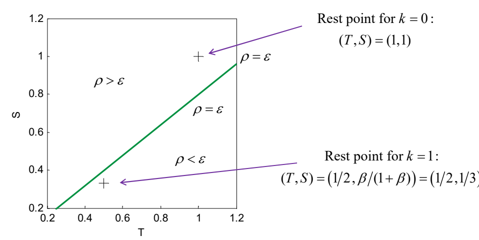
Ocean Circulation

Welander's Model

$$\dot{T} = 1 - T - k(\rho)T$$

$$\dot{S} = \beta(1-S) - k(\rho)S \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$$

$$\rho = -\alpha T + S$$



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Ocean Circulation

Welander's Model

$$\dot{T} = 1 - T - k(\rho)T$$

$$\dot{S} = \beta(1-S) - k(\rho)S \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$$

$$\rho = -\alpha T + S$$

At $\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\rho = -\alpha + 1 = 1/5.$ At $\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix}$, $\rho = -\alpha/2 + 1/3 = -2/5 + 1/3 = -1/15.$

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Ocean Circulation

Welander's Model

$$\dot{T} = 1 - T - k(\rho)T$$

$$\dot{S} = \beta(1-S) - k(\rho)S \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$$

$$\rho = -\alpha T + S$$

At $\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\rho = -\alpha + 1 = 1/5.$ At $\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix}$, $\rho = -\alpha/2 + 1/3 = -2/5 + 1/3 = -1/15.$

Wait a minute!!

The computation yielding an equilibrium at $(T,S) = (1,1)$ is valid only if $\rho < \varepsilon$, but the density at that point is $\rho = 1/5$.

The computation yielding an equilibrium at $(T,S) = (1/2, 1/3)$ is valid only if $\rho > \varepsilon$, but the density at that point is $\rho = -1/15$.

If ε is small, neither condition holds.

No equilibrium points!

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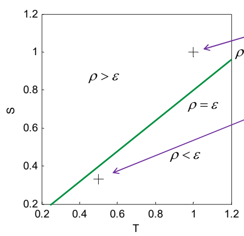
Ocean Circulation

Welander's Model

$$\dot{T} = 1 - T - k(\rho)T$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$$
$$\rho = -\alpha T + S$$



Rest point for $k = 0$:
$(T,S) = (1,1)$

$\boxed{\text{Valid only if } \varepsilon > 1/5.}$

Rest point for $k = 1$:
$(T,S) = (1/2, \beta/(1+\beta)) = (1/2, 1/3)$

$\boxed{\text{Valid only if } \varepsilon < -1/15.}$

Since ε is small, neither rest point is located where is is valid.

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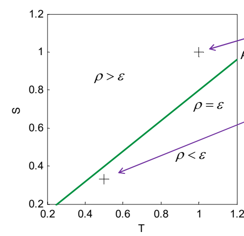
Ocean Circulation

Welander's Model

$$\dot{T} = 1 - T - k(\rho)T$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$$
$$\rho = -\alpha T + S$$



Rest point for $k = 0$:
$(T,S) = (1,1)$

$\boxed{\text{Valid only if } \varepsilon > 1/5.}$

Rest point for $k = 1$:
$(T,S) = (1/2, \beta/(1+\beta)) = (1/2, 1/3)$

$\boxed{\text{Valid only if } \varepsilon < -1/15.}$

Let's proceed anyway.

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Ocean Circulation

Welander's Model

$$\dot{T} = 1 - T - k(\rho)T$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases} \qquad \begin{matrix} \alpha = 0.8 \\ \beta = 0.5 \end{matrix}$$
$$\rho = -\alpha T + S$$

First consider the case $k(\rho) = 1$,
i.e., $\rho > \varepsilon$.

For $k(\rho) = 0$: $\quad \begin{aligned} \dot{T} &= 1 - 2T \\ \dot{S} &= \beta - (\beta+1)S \end{aligned}$

$$\frac{d}{dt}\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} 1 \\ \beta \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -(\beta+1) \end{bmatrix}\begin{bmatrix} T \\ S \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -3/2 \end{bmatrix}\begin{bmatrix} T \\ S \end{bmatrix}$$

What is the phase portrait for this equation?

Math 5490
Ocean Circulation

Welander's Model

$$\dot{T} = 1 - T - k(\rho)T$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases} \qquad \begin{matrix} \alpha = 0.8 \\ \beta = 0.5 \end{matrix}$$
$$\rho = -\alpha T + S$$

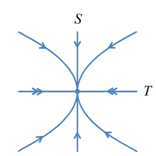
First consider the case $k(\rho) = 1$,
i.e., $\rho > \varepsilon$.

For $k(\rho) = 1$: $\quad \begin{aligned} \dot{T} &= 1 - 2T \\ \dot{S} &= \beta - (\beta+1)S \end{aligned}$

$$\frac{d}{dt}\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} 1 \\ \beta \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -(\beta+1) \end{bmatrix}\begin{bmatrix} T \\ S \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -3/2 \end{bmatrix}\begin{bmatrix} T \\ S \end{bmatrix}$$

Since T decreases to equilibrium faster than S, the decay to equilibrium looks like:



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Ocean Circulation

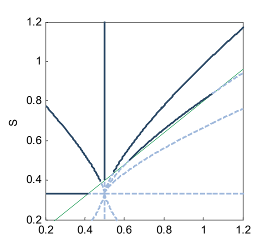
Welander's Model

$$\dot{T} = 1 - T - k(\rho)T$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases} \qquad \begin{matrix} \alpha = 0.8 \\ \beta = 0.5 \end{matrix}$$
$$\rho = -\alpha T + S$$

First consider the case $k(\rho) = 1$,
i.e., $\rho > \varepsilon$.

$$\frac{d}{dt}\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -3/2 \end{bmatrix}\begin{bmatrix} T \\ S \end{bmatrix}$$

Actually, it looks like:



The solid lines are above the the line $\rho = \varepsilon$, where the equation is valid, while the dotted lines show where the equation is invalid.

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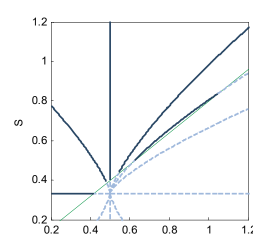
Ocean Circulation

Welander's Model

$$\dot{T} = 1 - T - k(\rho)T$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases} \qquad \begin{matrix} \alpha = 0.8 \\ \beta = 0.5 \end{matrix}$$
$$\rho = -\alpha T + S$$

Starting above the discontinuity line $\rho = \varepsilon$, solutions of Welander's model try to asymptotically approach the equilibrium at $(1/2, 1/3)$. Before they arrive, they hit the discontinuity line, and the system changes to the case $k(\rho) = 1$.



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Ocean Circulation

Welander's Model

$$\dot{T} = 1 - T - k(\rho)T \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases} \qquad \alpha = 0.8$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad \beta = 0.5$$
$$\rho = -\alpha T + S$$

Now consider the case $k(\rho) = 0$,
i.e., $\rho < \varepsilon$.

For $k(\rho) = 0$: $\dot{T} = 1 - T$, $\dot{S} = \beta - \beta S$

$$\frac{d}{dt}\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} 1 \\ \beta \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -\beta \end{bmatrix}\begin{bmatrix} T \\ S \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1/2 \end{bmatrix}\begin{bmatrix} T \\ S \end{bmatrix}$$

What is the phase portrait for this equation?

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Ocean Circulation

Welander's Model

$$\dot{T} = 1 - T - k(\rho)T \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases} \qquad \alpha = 0.8$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad \beta = 0.5$$
$$\rho = -\alpha T + S$$

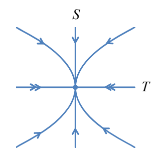
Now consider the case $k(\rho) = 0$,
i.e., $\rho < \varepsilon$.

For $k(\rho) = 0$: $\dot{T} = 1 - T$, $\dot{S} = \beta - \beta S$

$$\frac{d}{dt}\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} 1 \\ \beta \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -\beta \end{bmatrix}\begin{bmatrix} T \\ S \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1/2 \end{bmatrix}\begin{bmatrix} T \\ S \end{bmatrix}$$

As before, T decreases to equilibrium faster than S, so the decay to equilibrium looks like:



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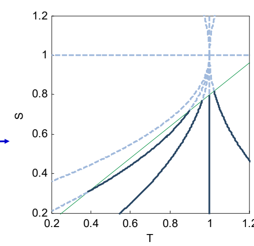
Welander's Model

$$\dot{T} = 1 - T - k(\rho)T \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases} \qquad \alpha = 0.8$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad \beta = 0.5$$
$$\rho = -\alpha T + S$$

Now consider the case $k(\rho) = 0$,
i.e., $\rho < \varepsilon$.

$$\frac{d}{dt}\begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1/2 \end{bmatrix}\begin{bmatrix} T \\ S \end{bmatrix}$$

Actually, it looks like:



The solid lines are blow the the line $\rho = \varepsilon$, where the above equation is valid, while the dotted lines show where the equation is invalid.

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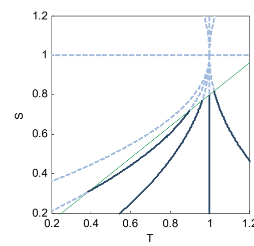
Ocean Circulation

Welander's Model

$$\dot{T} = 1 - T - k(\rho)T \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases} \qquad \alpha = 0.8$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad \beta = 0.5$$
$$\rho = -\alpha T + S$$

Starting below the discontinuity line $\rho = \varepsilon$, solutions of Welander's model try to asymptotically approach the equilibrium at $(1,1)$. Before they arrive, they hit the discontinuity line, and the system changes back to the case $k(\rho) = 1$.



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Ocean Circulation

Welander's Model

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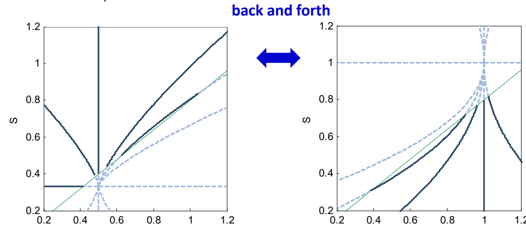
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad \beta = 0.5$$
$$\rho = -\alpha T + S$$

back and forth



Math 5490
Ocean Circulation

Welander's Model

$$\dot{T} = 1 - T - k(\rho)T \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases} \qquad \alpha = 0.8$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad \beta = 0.5$$
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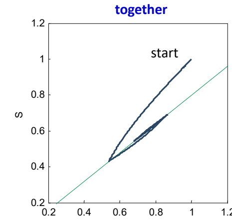
together



narrative

The temperature and salinity both start above the line. The shallow ocean density is higher than the deep ocean density, so an overturning circulation decreases both the temperature and salinity of the shallow ocean, bringing the density down until the system hits the discontinuity line, at which point the overturning circulation stops, and the temperature and salinity then start moving toward that of the atmosphere, until they hit the discontinuity line.

Repeat.

Math 5490
Ocean Circulation

Welander's Model

$$\dot{T} = 1 - T - k(\rho)T \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases} \qquad \alpha = 0.8$$
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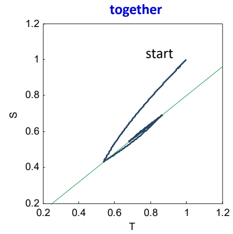

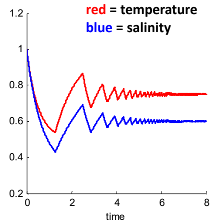

Math 5490
Ocean Circulation

Welander's Model

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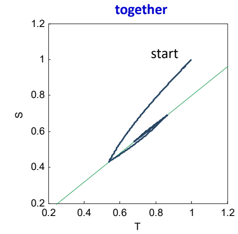

narrative

The temperature and salinity both start above the line. The shallow ocean density is higher than the deep ocean density, so an overturning circulation decreases both the temperature and salinity of the shallow ocean, bringing the density down until the system hits the discontinuity line, at which point the overturning circulation stops, and the temperature and salinity then start moving toward that of the atmosphere, until they hit the discontinuity line.

Repeat.

Math 5490
Ocean Circulation

Welander's Model

$$\dot{T} = 1 - T - k(\rho)T \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases} \qquad \alpha = 0.8$$
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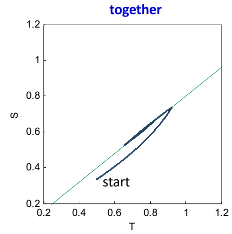
together



narrative

The temperature and salinity both start below the line. The shallow ocean density is lower than the deep ocean density, so there is no overturning circulation. Instead, the temperature and salinity both increase as they move toward equilibrium with the atmosphere. Before the equilibrium is reached, the system hits the discontinuity line and the density become high enough so that the overturning circulation starts and continues until the density becomes so low that the circulation stops.

Repeat.

Math 5490
Ocean Circulation

Welander's Model

$$\dot{T} = 1 - T - k(\rho)T \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases} \qquad \alpha = 0.8$$
$$\dot{S} = \beta(1 - S) - k(\rho)S \qquad \beta = 0.5$$
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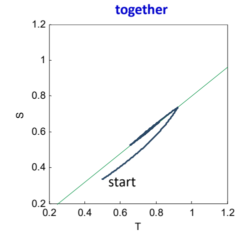
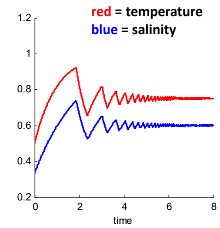
together





Math 5490
Ocean Circulation

Welander's Model

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$$\rho = -\alpha T + S$$

The details are easier to see if we introduce a salinity anomaly measuring the deviation of the salinity from that along the discontinuity line.

$$\Delta S = S - \alpha T - \varepsilon = \rho - \varepsilon$$

Note that this anomaly also measures the density of the sytem above the critical density ε.

Math 5490
Ocean Circulation

Welander's Model

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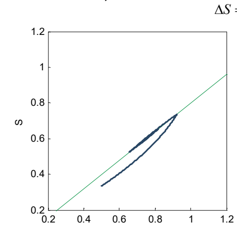
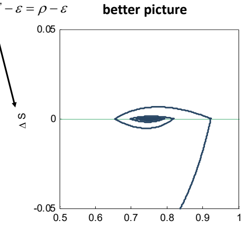
$$\dot{S} = \beta(1 - S) - k(\rho)S \qquad \beta = 0.5$$
$$\rho = -\alpha T + S$$

$\Delta S = S - \alpha T - \varepsilon = \rho - \varepsilon$ **better picture**





Math 5490
Ocean Circulation

Welander's Model

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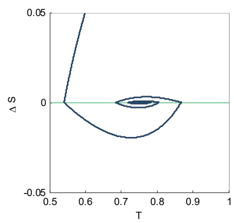
$$\rho = -\alpha T + S$$

$\boxed{\varepsilon = 0.002}$



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Math 5490
Ocean Circulation

Welander's Model

$$\dot{T} = 1 - T - k(\rho)T \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases} \qquad \alpha = 0.8$$
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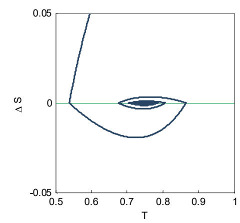
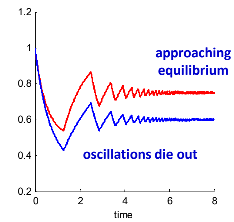
$$\rho = -\alpha T + S$$

$\boxed{\varepsilon = 0}$





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Math 5490
Ocean Circulation

Welander's Model

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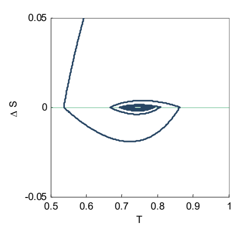
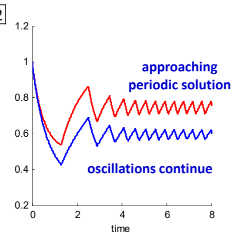
$\boxed{\varepsilon = -0.002}$





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Math 5490
Ocean Circulation

Welander's Model

$\boxed{\varepsilon = -0.02}$

$$\dot{T} = 1 - T - k(\rho)T \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases} \qquad \alpha = 0.8$$
$$\dot{S} = \beta(1 - S) - k(\rho)S \qquad \beta = 0.5$$
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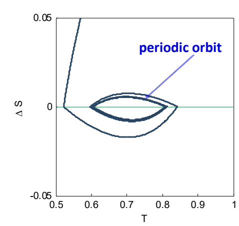
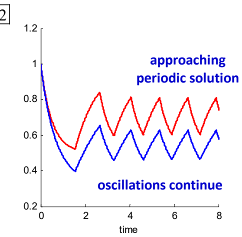
$\boxed{\varepsilon = -0.02}$





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Math 5490
Ocean Circulation

Welander's Model

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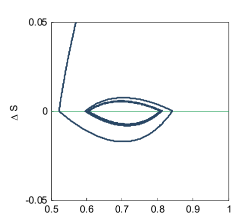
$\boxed{\varepsilon = -0.02}$



Note that the solution started outside the periodic orbit and spiraled down to it. The amplitude of the oscillations decrease asymptotically to that of the periodic orbit.

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Math 5490
Ocean Circulation

Welander's Model

$$\dot{T} = 1 - T - k(\rho)T \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases} \qquad \alpha = 0.8$$
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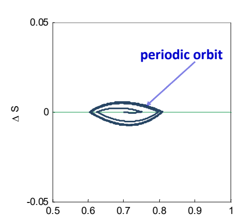

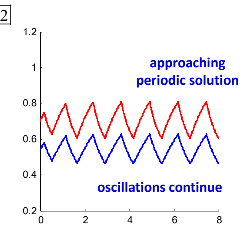

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Math 5490
Ocean Circulation

Welander's Model

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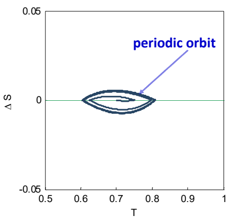


Note that this solution started inside the periodic orbit and spiraled out to it. The amplitude of the oscillations increase asymptotically to that of the periodic orbit.

This periodic orbit appears to be stable.

Math 5490
Ocean Circulation

What caused the Dansgaard-Oeschger oscillations?

They could be self-oscillations in the natural dynamics of ocean circulation.

Welander constructed a simple (*conceptual!*) box model of ocean circulation and showed that the interactions of temperature and salinity with the atmosphere, the surface ocean, and the deep ocean could create self-oscillations.

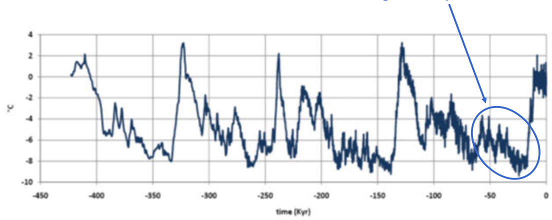
Pierre Welander, A simple heat-salt oscillator, *Dynamics of Atmospheres and Oceans* **6** (1982) 233-242.

Math 5490
Ocean Circulation

Recent (last 400 Kyr) Temperature Cycles

Vostok Ice Core Data

Explained by Welander?



J.R. Petit, *et al* (1999) Climate and atmospheric history of the past 420,000 years from the Vostok ice core, Antarctica, *Nature* 399, 429-436.

Math 5490
Ocean Circulation

Welander's Model

Mathematical Note

The mathematical tools to actually *prove* that Welander's model behaves in the way he described were not fully developed in 1982.

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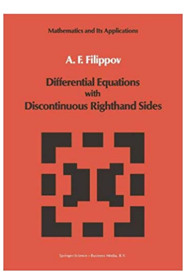
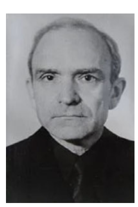
Ocean Circulation



Kluwer Academic Publishers

Published in Russian in 1985.



A.F. Filippov*

*https://alchetron.com/Aleksei-Fedorovich-Filippov

Math 5490
Ocean Circulation

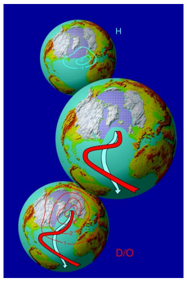
Mathematical Note

Welander assumed that the self-oscillations he found in his discontinuous model would hold held for a nearby smooth system.

Juliann Leifeld, PhD 2016:

Welander's assumption was correct.

Math 5490
Ocean Circulation



$$\frac{dT}{dt} = 1 - T - k(\rho)T$$
$$\frac{dS}{dt} = \beta(1 - S) - k(\rho)S$$
$$\rho = -\alpha T + S$$

Mountain Avens
(*Dryas octopetala*)

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