Math 1142
October 17, 2006

Optimization

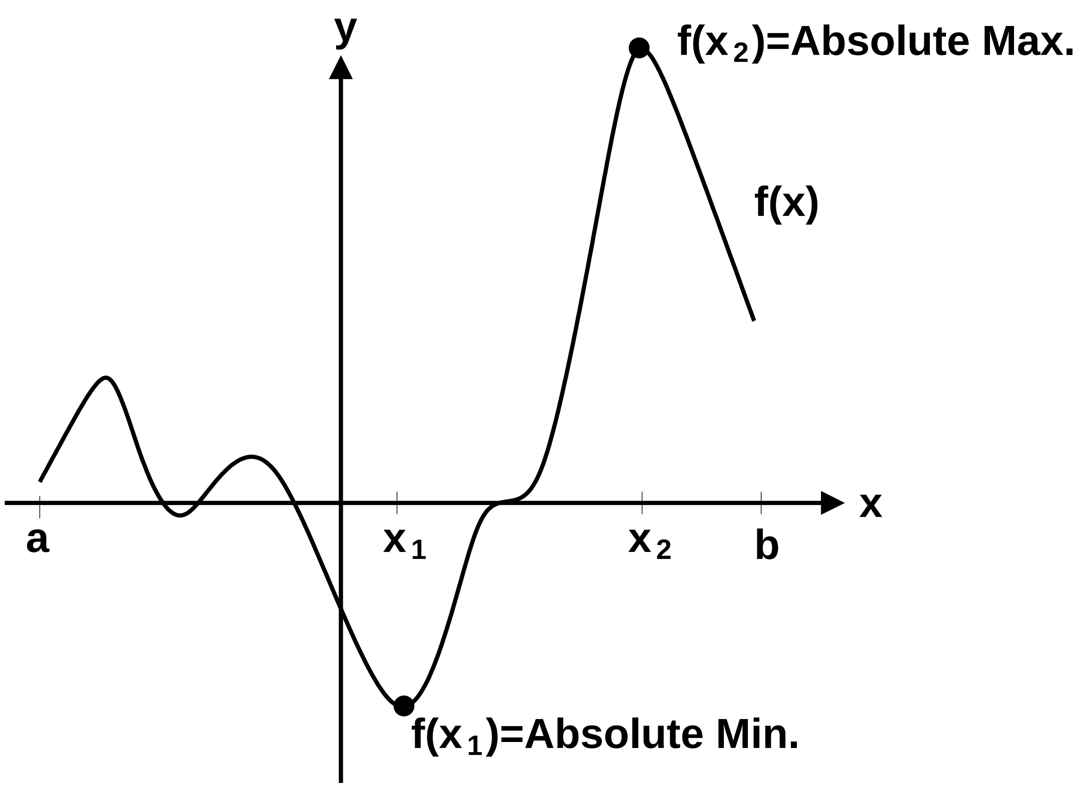
Optimization problems are concerned with finding the largest or smallest value of a function on an interval.

Examples:

- Maximizing profit.
- Minimizing energy consumption.
- Minimizing cost.

The largest value attained by a function on an interval is called its absolute maximum on that interval, while the smallest value attained by a function on an interval is called its absolute minimum on that interval.

Example: On the interval $a \leq x \leq b$, the absolute maximum of $f(x)$ is $y = f(x_2)$, while the absolute minimum of $f(x)$ is $y = f(x_1)$.



Let's quickly review critical numbers and the different types of intervals before turning to the procedure for finding absolute maxima and minima.

Critical Number: A number c in the domain of $f(x)$ such that $\underline{f'(c) = 0}$ or $\underline{f'(c) \text{ does not exist}}$.

Example: Find the critical numbers of $f(x) = \sqrt{x} - 2x$.

Rewrite $f(x) = x^{\frac{1}{2}} - 2x$. Then

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} - 2 = \frac{1}{2}\frac{1}{\sqrt{x}} - 2 = \frac{1}{2\sqrt{x}} - 2\frac{2\sqrt{x}}{2\sqrt{x}} = \frac{1-4\sqrt{x}}{2\sqrt{x}}$.

$f'(x)$ is undefined when its denominator is 0. Set the denominator equal to 0 and solve for x:

$$\begin{aligned} 2\sqrt{x} &= 0 \\ \sqrt{x} &= \frac{0}{2} = 0 \\ (\sqrt{x})^2 &= 0^2 \\ x &= 0. \end{aligned}$$

$f'(x) = 0$ when its numerator is 0. Set the numerator equal to 0 and solve for x:

$$\begin{aligned} 1 - 4\sqrt{x} &= 0 \\ 1 &= 4\sqrt{x} \\ \frac{1}{4} &= \sqrt{x} \\ \left(\frac{1}{4}\right)^2 &= \left(\sqrt{x}\right)^2 \\ \frac{1}{16} &= x. \end{aligned}$$

$$\boxed{\text{Critical Numbers: } x = 0 \text{ and } x = \tfrac{1}{16}}$$

Types of Intervals

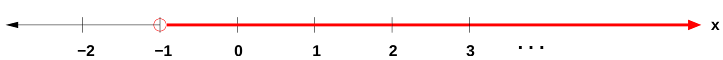
Closed Intervals:

- Are denoted $[a, b]$.
- Have the form $a \leq x \leq b$.
- Contain both endpoints.
- Example: The closed interval $[-1, 2]$.

Open Intervals:

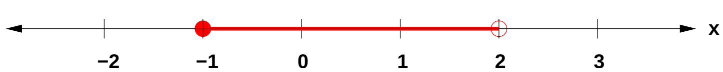
- Are denoted (a, b).
- Have the form $a < x < b$.
- Contain neither endpoint.
- Example 1: The open interval $(-1, 2)$.

- Example 2: The open interval $(-1, \infty)$ i.e. $x > -1$.

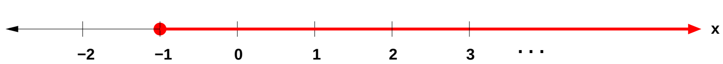


Half-Open Intervals:

- Are denoted $[a, b)$ or $(a, b]$.
- Have the form $a \leq x < b$ or $a < x \leq b$.
- Contain exactly one endpoint.
- Example 1: The half-open interval $[-1, 2)$.



- Example 2: The half-open interval $[-1, \infty)$ i.e. $x \geq -1$.



On Friday 10/13, we discussed a method for finding the absolute max. and min. on a closed interval. This method can be extended to include all types of intervals:

Procedure for Finding the Absolute Max. & Min. of a Continuous Function f on an Interval I *The interval I can be of the form $[a,b]$, (a,b), $[a,b)$, or $(a,b]$.*

Step 1: Find the critical numbers of f in the interval $a < x < b$.

Step 2:

☐ Critical Numbers: Evaluate $f(x)$ at the critical numbers found in Step 1.

☐ End Points: Evaluate $f(x)$ at any closed endpoints and take the limit of $f(x)$ as x approaches any open endpoints. Specifically,

- If $I = [a,b]$, find $f(a)$ and $f(b)$.
- If $I = (a,b)$, find $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$
- If $I = (a,b]$, find $\lim_{x\to a^+} f(x)$ and $f(b)$.
- If $I = [a,b)$, find $f(a)$ and $\lim_{x\to b^-} f(x)$.

Step 3:

If the largest value occurs at an open endpoint, then the absolute maximum does not exist. Otherwise, the largest value in Step 2 is the absolute maximum of $f(x)$ on the interval.

Similarly, if the smallest value occurs at an open endpoint, then the absolute minimum does not exist. Otherwise, the smallest value in Step 2 is the absolute minimum of $f(x)$ on the interval.

Example with an Open Interval:

Find the absolute maximum and minimum of $f(x) = x + \frac{4}{x^2}$ on $(0,4)$ and find where they are attained.

$f(x)$ is continuous on $0 < x < 4$ since its only discontinuity occurs at $x = 0$.

Step 1: Rewrite $f(x) = x + \frac{4}{x^2} = x + 4x^{-2}$. Then

$f'(x) = 1 + 4(-2)x^{-3} = 1 - \frac{8}{x^3} = \frac{x^3}{x^3} - \frac{8}{x^3} = \frac{x^3-8}{x^3}$.

$f'(x)$ is always defined on $0 < x < 4$ (since 0 doesn't lie in this interval).

$f'(x) = 0$ when its numerator is 0: $x^3 - 8 = 0 \;\Rightarrow\; x^3 = 8 \;\Rightarrow\; x = 2$.

So the only critical number is $x = 2$.

Step 2:

☐ Critical Numbers: Evaluate $f(2) = 2 + \frac{4}{2^2} = \underline{3}$.

☐ End Points: Since $(0,4)$ is an open interval, calculate:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(x + \frac{4}{x^2}\right) = 0 + \infty = \underline{+\infty}$$

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-}\left(x + \frac{4}{x^2}\right) = 4 + \frac{4}{4^2} = \underline{4\frac{1}{4}}$$

Step 3:

The largest value in step 2 is $+\infty$ and this occurs at the open endpoint $x = 0$.

$\boxed{\text{So } f(x) \text{ has no absolute max. on this interval.}}$

The smallest value in step 2 is 3 and this occurs at $x = 2$, which is in the interval $(0, 4)$.

$\boxed{\text{So the absolute min. on this interval is 3 and is attained at } x = 2.}$

Example with a Half-Open Interval:

Find the absolute maximum and minimum of $f(x) = 2x(x+1)^4$ on $(-2, -\frac{1}{2}]$ and find where they are attained.

$f(x)$ is continuous on $-2 < x \leq -\frac{1}{2}$.

Step 1: Calculate $f'(x)$:

$$\begin{aligned} f'(x) &= 2x \cdot \left((x+1)^4\right)' + (2x)' \cdot (x+1)^4 \text{ by the product rule} \\ &= 2x \cdot \left(4(x+1)^3 \cdot 1\right) + 2 \cdot (x+1)^4 \\ &= 8x(x+1)^3 + 2(x+1)^4 \\ &= 2(x+1)^3\left(4x + (x+1)\right) \text{ by factoring } 2(x+1)^3 \text{ out of each term} \\ &= 2(x+1)^3(5x+1) \end{aligned}$$

$f'(x)$ is always defined.

$f'(x) = 0$ when $x + 1 = 0$ or when $5x + 1 = 0$; i.e. when $x = -1$ or when $x = -\frac{1}{5}$.

$-\frac{1}{5}$ doesn't lie in the interval $-2 < x \leq -\frac{1}{2}$.
So the only critical number is $x = -1$.

Step 2:

☐ Critical Numbers: Evaluate $f(-1) = 2(-1)(-1+1)^4 = \underline{0}$.

☐ End Points: Since the interval is half-open, evaluate $f(x)$ at the closed endpoint and take the limit at the open endpoint:

$f(-\frac{1}{2}) = 2\left(-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right)^4 = -1\left(\frac{1}{2}\right)^4 = \underline{-\frac{1}{16}}$.

$\lim_{x\to-2^+} f(x) = \lim_{x\to-2^+} 2x(x+1)^4 = 2(-2)(-2+1)^4 = \underline{-4}$.

Step 3:

The largest value in step 2 is 0 and this occurs at $x = -1$, which is in the interval $(-2, -\frac{1}{2}]$.

$\boxed{\text{So the absolute max. is 0 and is attained at } x = -1.}$

The smallest value in step 2 is -4 and this occurs at the open end point $x = -2$.

$\boxed{\text{So } f(x) \text{ has no absolute min. on this interval.}}$

Example with a Closed Interval:

Find the absolute maximum and minimum of $f(x) = 5 + x^2 + \frac{1}{x^2}$ on $[\frac{1}{4}, 2]$ and find where they are attained.

$f(x)$ is continuous on $\frac{1}{4} \le x \le 2$ since its only discontinuity occurs at $x = 0$.

Step 1: Rewrite $f(x) = 5 + x^2 + x^{-2}$. Then

$f'(x) = 0 + 2x + -2 \cdot x^{-3} = 2x + \frac{-2}{x^3} = \frac{2x^4}{x^3} + \frac{-2}{x^3} = \frac{2x^4-2}{x^3}$.

$f'(x)$ is always defined on $\frac{1}{4} \le x \le 2$ (since 0 doesn't belong to this interval).

$f'(x) = 0$ when the numerator is 0:

$$2x^4 - 2 = 2(x^4 - 1) = 2(x^2 + 1)(x^2 - 1) = 2(x^2 + 1)(x - 1)(x + 1) = 0 \quad \Rightarrow \quad x = 1, x = -1.$$

-1 is not in the interval $\frac{1}{4} \le x \le 2$.

So the only critical number is $x = 1$.

Step 2:

☐ Critical Numbers: Evaluate $f(1) = 5 + 1^2 + \frac{1}{1^2} = \underline{7}$.

☐ End Points: Since the interval is closed, evaluate $f(x)$ at both endpoints:

$f(\frac{1}{4}) = 5 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{\left(\frac{1}{4}\right)^2}\right) = 5 + \frac{1}{16} + 16 = \underline{21\frac{1}{16}}$

$f(2) = 5 + 2^2 + \frac{2}{2^2} = 5 + 4 + \frac{1}{4} = \underline{9\frac{1}{4}}$.

Step 3:

Absolute max. is $21\frac{1}{16}$ and is attained at $x = \frac{1}{4}$.

Absolute min. is 7 and is attained at $x = 1$.