How to find the Absolute Max. $\&$ Min. of a Continuous Function f on an Interval I

The interval I can be of the form $[a, b]$, (a, b), $[a, b)$, *or* $(a, b]$.

Step 1: Find the critical numbers of f in the interval $a < x < b$.

Step 2:

☐ Critical Numbers: Evaluate $f(x)$ at the critical numbers found in Step 1.

☐ End Points: Evaluate $f(x)$ at any closed endpoints and take the limit of $f(x)$ as x approaches any open endpoints. Specifically,

- If $I = [a, b]$, find $f(a)$ and $f(b)$.
- If $I = (a, b)$, find $\lim_{x \to a^+} f(x)$ and $\lim_{x \to b^-} f(x)$
- If $I = (a, b]$, find $\lim_{x \to a^+} f(x)$ and $f(b)$.
- If $I = [a, b)$, find $f(a)$ and $\lim_{x \to b^-} f(x)$.

Step 3:

If the largest value in Step 2 occurs at an open endpoint, then the absolute maximum does not exist.

Otherwise, the largest value in Step 2 is the absolute maximum of $f(x)$ on the interval.

Similarly, if the smallest value in Step 2 occurs at an open endpoint, then the absolute minimum does not exist.

Otherwise, the smallest value in Step 2 is the absolute minimum of $f(x)$ on the interval.

Find the Errors!

Example: Find the absolute maximum and minimum of $f(x) = 2x(x+1)^4$ on $(-2, -\frac{1}{2}]$ and find where they are attained.

Note $f(x)$ is continuous on $-2 < x \leq -\frac{1}{2}$.

Step 1: $f'(x) = 2(x+1)^3(5x+1)$

The critical numbers are $x = -1$ and $x = -\frac{1}{5}$.

Step 2:

☐ Critical Numbers: $f(-1) = \underline{0}$ and $f(-\frac{1}{5}) = \underline{-.16}$

☐ End Points: $f(-\frac{1}{2}) = \underline{-.0625}$ and $\lim_{x \to -2^+} f(x) = \underline{-4}$.

Step 3:

The absolute max. is 0 and is attained at $x = -1$.

The absolute min. is -4 and is attained at $x = -2$.

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The critical numbers are $x = -1$ and $x = -\frac{1}{5}$. *$-\frac{1}{5}$ doesn't lie in the interval*

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☐ Critical Numbers: $f(-1) = \underline{0}$ and $f(-\frac{1}{5}) = \underline{-.16}$

☐ End Points: $f(-\frac{1}{2}) = \underline{-.0625}$ and $\lim_{x \to -2^+} f(x) = \underline{-4}$.

Step 3:

The absolute max. is 0 and is attained at $x = -1$.

The absolute min. is -4 and is attained at $x = -2$. *The absolute min. does not exist*

Correct Solution

Example: Find the absolute maximum and minimum of $f(x) = 2x(x+1)^4$ on $(-2, -\frac{1}{2}]$ and find where they are attained.

Note $f(x)$ is continuous on $-2 < x \leq -\frac{1}{2}$.

Step 1: Calculate $f'(x)$:

$$\begin{aligned} f'(x) &= 2x \cdot \left((x+1)^4\right)' + (2x)' \cdot (x+1)^4 \\ &= 2x \cdot \left(4(x+1)^3 \cdot 1\right) + 2 \cdot (x+1)^4 \\ &= 8x(x+1)^3 + 2(x+1)^4 \\ &= 2(x+1)^3\,(4x + (x+1)) \\ &= 2(x+1)^3(5x+1) \end{aligned}$$

Reminder: $f'(x) = 2(x+1)^3(5x+1)$

$f'(x)$ is always defined.

$f'(x) = 0$ when $x = -1$ or when $x = -\frac{1}{5}$.

$x = -\frac{1}{5}$ doesn't lie in the interval $-2 < x \leq -\frac{1}{2}$.

The only critical number is $x = -1$.

Step 2:

☐ Critical Numbers: Evaluate $f(-1) = 2(-1)(-1+1)^4 = \underline{0}$.

☐ End Points: Since the interval is half-open, evaluate $f(x)$ at the closed endpoint and take the limit at the open endpoint:

$$f(-\tfrac{1}{2}) = 2\left(-\tfrac{1}{2}\right)\left(-\tfrac{1}{2}+1\right)^4 = -1\left(\tfrac{1}{2}\right)^4 = \underline{-\tfrac{1}{16}}.$$

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} 2x(x+1)^4 = 2(-2)(-2+1)^4 = \underline{-4}.$$

Step 3:

The largest value in step 2 is 0 and this occurs at $x = -1$, which is in the interval $(-2, -\frac{1}{2}]$.

So the absolute max. is 0 and is attained at $x = -1$.

The smallest value in step 2 is -4 and this occurs at the open end point $x = -2$.

So $f(x)$ has no absolute min. on this interval.