Math 1142
November 22, 2006

Lecture Notes

Topics for Today:

☐ Verifying solutions to differential equations.

☐ Examples from Section 6.2.

- Exponential Growth and Decay
- Learning Models
- Dilution Models

Verifying Solutions: Taking the derivative of the solution should give the original differential equation.

Example: On Monday, we found that

$$y = -\frac{1}{2}e^{-2x} + \ln|x| + \frac{1}{2}e^{-2}$$

is the particular to the differential equation

$$\frac{dy}{dx} = e^{-2x} + \frac{1}{x},$$

satisfying $y = 0$ when $x = 1$.

To verify that this is the solution, take the derivative of both sides:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(-\frac{1}{2}e^{-2x} + \ln|x| + \frac{1}{2}e^{-2}\right) \\
\frac{dy}{dx} &= -\frac{1}{2}e^{-2x}(-2) + \frac{1}{x} + 0 \\
\frac{dy}{dx} &= -\frac{1}{\cancel{2}}e^{-2x}(-\cancel{2}) + \frac{1}{x} \\
\frac{dy}{dx} &= e^{-2x} + \frac{1}{x}.
\end{aligned}$$

This last line is the original differential equation, which verifies that the solution was correct.

Examples from Section 6.2:

A quick reminder (from Section 1.4, page 46):

A quantity Q is *directly proportional* (or *proportional*) to x if $Q = kx$ for some constant k.

A quantity Q is *jointly proportional* to x and y if $Q = kxy$ for some constant k.

Exponential Growth and Decay

Recall from Chapter 4:

A quantity $Q(t)$ grows exponentially if $Q(t) = Q_0 e^{kt}$ with $k > 0$.

A quantity $Q(t)$ decays exponentially if $Q(t) = Q_0 e^{-kt}$ with $k > 0$.

In both cases, $\frac{dQ}{dt} = mQ$ for some constant m, i.e. the rate of change of Q is proportional to Q.

The converse is also true:

If $\frac{dQ}{dt} = mQ$, then $Q(t) = Q_0 e^{mt}$ (where $Q_0 = Q(0)$).

In words:

If the rate of change of Q is proportional to Q, then Q grows exponentially (if $m > 0$) or decays exponentially (if $m < 0$).

This is proved in Example 6.2.6 (pages 483-484).

Learning Models

If a quantity Q grows at a rate proportional to the difference between its size and a fixed upper limit B (i.e. if $\frac{dQ}{dt} = k(B - Q)$ for some constant k), then Q can be represented as $Q(t) = B - Ae^{-kt}$.

These are called learning models because functions of the form $Q(t) = B - Ae^{-kt}$ can be used to describe the relationship between worker efficiency and the amount of training or experience the worker has. The graphs of these functions are called learning curves. These models can also be used in other situations, as in the next example.

Example: Problem 47 (page 494)

After being placed in a container of water, sugar dissolves at a rate proportional to the amount of undissolved sugar remaining in the container. Express the amount of sugar that has been dissolved as a function of time and draw the graph.

Variables: S = the amount of sugar dissolved, t = time, S_0 = the amount of sugar initially placed in the container.

Goal: Find an equation for $S(t)$.

Equations: The amount of undissolved sugar in the container is $S_0 - S$. The phrase "the sugar dissolves at a rate proportional to the amount of undissolved sugar remaining in the container" can be written mathematically as

$$\frac{dS}{dt} = k(S_0 - S) \text{ for some constant } k.$$

To find $S(t)$, solve this differential equation using separation of variables, i.e. separate the variables and integrate both sides:

$$\begin{aligned}
\frac{dS}{dt} &= k(S_0 - S) \\
\int \frac{dS}{S_0 - S} &= \int k dt \text{ (For the integral on the left, you can set } u = S_0 - S \text{ and } du = -dS.\text{)} \\
-\ln|S_0 - S| + C_1 &= kt + C_2 \\
-\ln(S_0 - S) &= kt + C_2 - C_1 \text{ (The absolute values are dropped since } S_0 > S.\text{)} \\
-\ln(S_0 - S) &= kt + C \text{ by setting } C = C_2 - C_1 \\
\ln(S_0 - S) &= -kt - C \\
e^{\ln(S_0 - S)} &= e^{-kt-C} \\
S_0 - S &= e^{-kt}e^{-C} \\
S_0 - S &= Ae^{-kt} \text{ by setting } A = e^{-C} \\
-S &= -S_0 + Ae^{-kt} \\
S &= S_0 - Ae^{-kt}
\end{aligned}$$

$$\underline{S(t) = S_0 - Ae^{-kt}}$$

Next find A:

Setting $t = 0$ in the equation gives: $S(0) = S_0 - Ae^{k\cdot 0} = S_0 - A.$

When $t = 0$, no sugar has dissolved yet, so $S(0) = 0$.

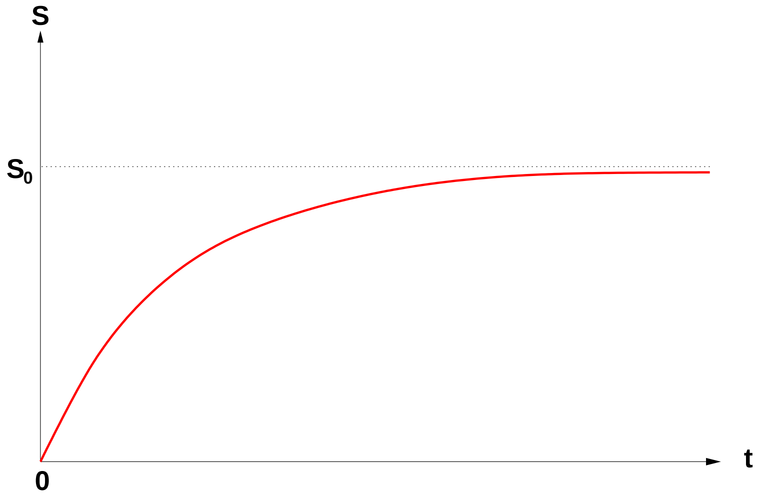
Setting these two expressions for $S(0)$ equal gives: $S_0 - A = 0$, i.e. $S_0 = A$.

Plugging $S_0 = A$ into the equation above gives:

$$\underline{S(t) = S_0 - S_0e^{-kt} = S_0(1 - e^{-kt}).}$$

(*Please note that there is not enough information given in the problem to solve for k. However, we do know that k must be positive since the amount of dissolved sugar always increases. That is, $\frac{dS}{dt}$ must be positive.*)

The graph of $S(t)$ is:



Dilution Models

I began Example 6.2.9 (page 489) and will finish this on Monday. It may be helpful to read through the set-up and first steps before class.