1.1 Let $A=\{a_1,\dots,a_m\}$; since f is 1-to-1, & all the a_i's are distinct, the $f(a_i)$'s are distinct (more explicitly, $f(a_i)=f(a_j)$ iff $a_i=a_j$ (by 1-to-1)). Hence $f(A):\overset{\text{def}}{=}\{f(a_1),f(a_2),\dots,f(a_m)\}\subseteq B$ has m distinct elements, & since a set always has at least as many elements as any one of its subsets, we have

$$n=\#(B)\geq\#(f(A))=m$$

1.2 If $f:A\to B$ is bijective, then it is injective, hence by (1.1) $\boxed{m\leq n}$. Moreover, if $f:A\to B$ is bijective, then f admits an inverse map $f^{-1}:B\longrightarrow A$ which is also bijective. Since f^{-1} is bijective, it is injective; hence, applying (1.1) again (taking f^{-1} in place of f, & switching the roles of A & B), we get an inequality:

$$n=\#(B)\leq\#(A)=m \quad (\text{i.e, } \boxed{n\leq m}).$$

Combining the boxed inequalities, we get $n=m$.

2.1 A function $f:A\to B$ is uniquely determined by listing the values $f(a)\in B$, for each $a\in A$. In other words, one can find all the functions $\{a_1,a_2,a_3\}\to\{1,2,3\}$ by determining all ways to assign a number 1, 2 or 3 to a_1, then assign a number 1, 2 or 3 to a_2, & then finally assign a number 1, 2 or 3 to a_3. There are three ways to make such an assignment for a_1,

——————— // ——————— a_2, and

——————— // ——————— a_3. Since

the assignment for a_1 doesn't affect the assignment for a_2, there is a total $3\cdot3=9$ ways to make an assignment for a_1 & then make an assigment for a_2. The assignment for a_1 & a_2 don't affect the assignment for a_3, there is

2.1 cont.)

a total of $9\cdot 3=27$ ways to assign a_1 & a_2, and then assign a_3. Thus, there are $\underline{\underline{27}}$ functions $\{a_1,a_2,a_3\}\longrightarrow\{1,2,3\}$.

2.2 We follow the same process as 2.1, but in this case, we must make a total of four assignments, and each assignment allows only two choices. Thus we get

$$\boxed{\underset{\substack{\uparrow\\ \text{two choices}\\ \text{for } a_1}}{2}\cdot\overset{\substack{\text{two choices for } a_2\\ \downarrow}}{2}\cdot\underset{\substack{\uparrow\\ \text{two choices}\\ \text{for } a_3}}{2}\cdot\overset{\substack{\text{two choices for } a_4\\ \swarrow}}{2}=\boxed{16}}$$

functions $\{a_1,a_2,a_3,a_4\}\longrightarrow\{0,1\}$.

2.3 Careful wording allows us to generalize the specific cases above to arbitrary finite sets A & B. I.e, if $n=\#(B)$ & $m=\#(A)$, then to define a function, for each $a\in A$ we must [redacted] assign an element of B. For each fixed $a\in A$ there are $n=\#(B)$ many choices for this assignment, thus to define a function we must make $m=\#(A)$ independent assignments (i.e, the choice made for a given element of A does not affect the assignments of other elements), & each assignment can be done in $n=\#(B)$ many ways. Hence, there are

$$\underbrace{n\cdot n\cdot\ldots\cdot n}_{m\text{-times (one for each element of } A)}=\boxed{n^m}$$

functions $A\longrightarrow B$.

3.1

$f(1,1)=1$; $f(2,1)=2$; $f(1,2)=3$; $f(1,3)=6$; $f(2,2)=5$; $f(3,1)=4$

3.2 a) $(n+m)=(n'+m')$.

If $f(n,m)=f(n',m')$ — i.e, $\frac{1}{2}(n+m-2)(n+m-1)+m=\frac{1}{2}(n'+m'-2)(n'+m'-1)+m'$, then

$\frac{1}{2}\left[(n+m-2)(n+m-1)-(n'+m'-2)(n'+m'-1)\right]=m'-m$. But $(n+m)=(n'+m')$, so the left-hand side of this equation must be zero. Hence $0=m'-m$, i.e, $\boxed{m=m'}$. Now, since $(n+m)=(n'+m')$ & $m=m'$, it follows immediately that $\boxed{n=n'}$ (substitute m' for m in $(n+m)=(n'+m')$ & solve for n). Thus, (under assumption $(n+m)=(n'+m')$), $f(n,m)=f(n',m') \Rightarrow (m=m'$ & $n=n') \Rightarrow (n,m)=(n',m')$.

3.2 b) Assume $n=n'$ & $n+m\neq n'+m'$, Suppose $f(n,m)=f(n',m')$, then we have

$\frac{1}{2}(n+m-2)(n+m-1)+m=\frac{1}{2}(n'+m'-2)(n'+m'-1)+m'$

$\underline{(n+m-2)(n+m-1)-(n'+m'-2)(n'+m'-1)=2(m'-m)}$

Mult 2
Subtract $(n'+m'-2)(n'+m'-1)$
Subtract $2m$

Now, since $n+m\neq n'+m'$, either $n+m \lneq n'+m'$ or else $n+m \gneq n'+m'$.

If $n+m \lneq n'+m'$, then (since $n=n'$) $m \lneq m'$ so the right-hand side of the underlined equation is strictly positive. Since $n+m \lneq n'+m'$, however, the left-hand side must be strictly negative. $[0\leq n+m-2 \lneq n'+m'-2$ & $0\leq n+m-1 \lneq n'+m'-1$ imply that $(n+m-2)(n+m-1) \lneq (n'+m'-2)(n'+m'-1)]$. Thus, we have reached a contradiction (it's impossible for a strictly positive number to be equal to a strictly negative number). It follows then that under assumption b), $f(n,m)=f(n',m')$ is FALSE, thus $\underline{f(n,m)=f(n',m') \Rightarrow (n,m)=(n',m')}$ is TRUE.

If $n+m \gneq n'+m'$, then we deduce that the RHS is strictly negative, but the LHS is strictly positive, using identical reasoning.

3.2 c) Assume $m=m'$ & $n+m \neq n'+m'$. Suppose $f(n,m)=f(n',m')$, then

$$\frac{1}{2}(n+m-2)(n+m-1)+m = \frac{1}{2}(n'+m'-2)(n'+m'-1)+m'$$

$$\underline{(n+m-2)(n+m-1)-(n'+m'-2)(n'+m'-1) = 2(m'-m) = 0}$$

since $m'=m$

Now since $n+m \neq n'+m'$, we have either $n+m \lneq n'+m'$ or else $n+m \gneq n'+m'$. In either case, using reasoning similar to b), we get that $(n+m-2)(n+m-1)-(n'+m'-2)(n'+m'-1) \neq 0$ (as it will be strictly positive or strictly negative). Thus $f(n,m)=f(n',m')$ leads to a contradiction (under assumption c)), & hence $f(n,m)=f(n',m')$ is <u>FALSE</u>, so

<u>$f(n,m)=f(n',m') \Rightarrow (n,m)=(n',m')$</u> is <u>TRUE</u>.

3.3 [redacted] $f(3,3)=13$, $f(3,4)=19$, $f(1,7)=28$

3.4 $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is the inverse of the map that enumerates $\mathbb{N} \times \mathbb{N}$ as indicated in the picture