Professional Problem #3 Solutions

1. a) Claim: $\forall\, n \geq 2$, $3^n - 2^n \geq 2^n$.

<u>Pf of claim</u>: When $n=2$ we have $3^2 - 2^2 = 5 \geq 4 = 2^2$.

Now assume for some $n \geq 2$ that $3^n - 2^n \geq 2^n$.

Then
$$3^{n+1} - 2^{n+1} = 3 \cdot 3^n - 2 \cdot 2^n$$
$$\geq 2(3^n - 2^n)$$
$$\geq 2 \cdot 2^n = 2^{n+1}.$$

Thus the claim is true by induction.

Well $3^n - 2^n \geq 2^n \Rightarrow \frac{1}{3^n - 2^n} \leq \frac{1}{2^n}$.

We can see that $\sum\limits_{n=2}^{\infty} \frac{1}{2^n} = \sum\limits_{n=2}^{\infty} \left(\frac{1}{2}\right)^n$ is a convergent geometric series since $\left|\frac{1}{2}\right| < 1$. Since $\forall\, n \geq 2$

$0 \leq \frac{1}{3^n - 2^n} \leq \frac{1}{2^n}$, we have that $\sum\limits_{n=2}^{\infty} \frac{1}{3^n - 2^n}$ is convergent

by the comparison theorem.

It follows that $\sum\limits_{n=1}^{\infty} \frac{1}{3^n - 2^n}$ is <u>convergent</u> since

$$\sum_{n=1}^{\infty} \frac{1}{3^n - 2^n} = 1 + \sum_{n=2}^{\infty} \frac{1}{3^n - 2^n}.$$

b) Observe that
$$\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n^2 - \sqrt{n}}} = \lim_{n\to\infty} \frac{n^2 - \sqrt{n}}{n^2} = \lim_{n\to\infty} 1 - \frac{\sqrt{n}}{n^2} = 1 > 0.$$

By the limit comparison test we have that
$$\sum_{n=2}^{\infty} \frac{1}{n^2 - \sqrt{n}} \text{ is convergent} \Leftrightarrow \sum_{n=2}^{\infty} \frac{1}{n^2} \text{ is convergent.}$$

Indeed, $\sum\limits_{n=2}^{\infty} \frac{1}{n^2}$ <u>converges</u> by the p-test, so $\sum\limits_{n=2}^{\infty} \frac{1}{n^2 - \sqrt{n}}$ is convergent.

1\. c) We will show that $\sum_{n=1}^{\infty} \frac{2n+1}{\sqrt{n^4+1}}$ is <u>divergent</u> using the limit comparison test. Observe that

$$\lim_{n\to\infty} \frac{1/n}{\frac{2n+1}{\sqrt{n^4+1}}} = \lim_{n\to\infty} \frac{\sqrt{n^4+1}}{2n^2+n} = \lim_{n\to\infty} \frac{1/n^2}{1/n^2} \frac{\sqrt{n^4+1}}{2n^2+n}$$

$$= \lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{n^4}}}{2+\frac{1}{n}} = \frac{\sqrt{1}}{2} = \frac{1}{2} > 0.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, it follows that $\sum_{n=1}^{\infty} \frac{2n+1}{\sqrt{n^4+1}}$ is divergent.

2\. For both (1) & (2) the answer is "<u>not necessarily</u>". Consider the sequences defined by $a_n = \begin{cases} \frac{1}{n^2} & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$ & $b_n = \frac{1}{n^2}$.

Then $0 < a_{2n} \leq b_{2n}$ $\forall n$, & $\sum_{n=1}^{\infty} a_n$ is divergent while $\sum_{n=1}^{\infty} b_n$ is convergent.

This provides an immediate counter-example to answering (1) & (2) with "yes."

3. We can solve the integral first: $\forall n \in \mathbb{N}$

$$\int_n^{2n} \frac{1}{x^{2/3}}\,dx = -\frac{3}{4} x^{-4/3} \Big|_n^{2n} = -\frac{3}{4}\left(\frac{1}{(2n)^{4/3} - n^{4/3}}\right)$$

$$= -\frac{3}{4}\left(\frac{1}{(2^{4/3}-1)n^{4/3}}\right)$$

$$= -\frac{3}{4(2^{4/3}-1)} \cdot \frac{1}{n^{4/3}}.$$

Now, let $c = -\frac{3}{4(2^{4/3}-1)}$. (Note that c is just a constant).

We have now that

$$\sum_{n=1}^{\infty} \int_n^{2n} \frac{1}{x^{2/3}}\,dx = \sum_{n=1}^{\infty} c \cdot \frac{1}{n^{4/3}}.$$

By the p-test $\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$ is convergent thus $\sum_{n=1}^{\infty} c\, \frac{1}{n^{4/3}}$ is convergent by theorem 4.2.4(i).

4. a) Pf: We have that $a_n > 0$ & $b_n > 0$ for every n, so we can apply the limit comparison test.

Notice that

$$\lim_{n\to\infty} \frac{a_n b_n}{a_n} = \lim_{n\to\infty} b_n = 0$$

since $\sum b_n$ is a convergent series. Since $\sum a_n$ is convergent it follows that $\sum a_n b_n$ is convergent by theorem 4.4.6 (2).

b) Claim: If $\sum_{n=1}^{\infty} a_n$ is a positive series which converges then $\forall\, m \in \mathbb{N}$ $\sum_{n=1}^{\infty} a_n^m$ is a convergent series.

Pf of claim: (by induction).

Let $m=1$. Then $\sum_{n=1}^{\infty} a_n^1 = \sum_{n=1}^{\infty} a_n$ is a convergent series by assumption.

Now assume for some $m \in \mathbb{N}$ that $\sum_{n=1}^{\infty} a_n^m$ is a convergent series. Since $\forall\, n \in \mathbb{N}$ $a_n > 0$, we have that $\sum_{n=1}^{\infty} a_n^m$ is a positive convergent series. By part (a) above we have that since $\sum_{n=1}^{\infty} a_n^m$ & $\sum_{n=1}^{\infty} a_n$ are positive convergent series, the series

$$\sum_{n=1}^{\infty} a_n^m \cdot a_n = \sum_{n=1}^{\infty} a_n^{m+1}$$ is also convergent.

By induction, we have shown that $\sum_{n=1}^{\infty} a_n^m$ is convergent for every $m \in \mathbb{N}$, as desired.