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3283 NOTES

ROOT TEST & RATIO TEST

We have (so far) talked about tests for series convergence that rely on some kind of comparison (either to another series or to an integrable function). We would like to find some test that doesn't rely on an external information (i.e, doesn't require some knowledge about another series).

Thm: $\boxed{\text{The ratio test}}$ LET $\sum_{n=1}^{\infty} a_n$ be a series and asume that $a_n > 0$ $\forall n \in \mathbb{N}$; we define a sequence $\boxed{r_n := \frac{a_{n+1}}{a_n}}$ (sequence of successive ratios). Then

(i) If $\langle r_n \rangle$ converges to a number strictly less than 1, then $\sum_{n=1}^{\infty} a_n$ $\boxed{\text{converges}}$

(ii) If $\langle r_n \rangle$ converges to a number strictly greater than 1, then $\sum_{n=1}^{\infty} a_n$ $\boxed{\text{diverges}}$

(iii) If $\langle r_n \rangle$ converges to 1, or if $\langle r_n \rangle$ fails to converge, then the ratio test "fails" (that is to say $\sum a_n$ may converge or diverge & the ratio test doesn't tell us which).

~~(iv) If $\langle r_n \rangle$ is unbounded, then $\sum_{n=1}^{\infty}$~~

~~(v) If $\langle r_n \rangle$ diverges to ∞, then $\sum a_n$ diverges.~~

Proof: (i) If $r_n \longrightarrow L < 1$, then for all "large enough" (i.e, $\exists n_0$ s.t. $\forall n \geq n_0$) $r_n < \frac{L+1}{2} < 1$. Now, note that

$$a_{n_0} \cdot r_{n_0} \cdot r_{n_0+1} \cdots r_{n_0+K} = a_{n_0} \cdot \frac{a_{n_0+1}}{a_{n_0}} \cdot \frac{a_{n_0+2}}{a_{n_0+1}} \cdots \frac{a_{n_0+K+1}}{a_{n_0+K}} = a_{n_0+K+1}.$$

Hence

$$a_{(n_0+K+1)} = a_{n_0} \cdot (\underbrace{r_{n_0} \cdots r_{n_0+K}}_{K\text{-factors, each less than } \frac{L+1}{2}}) < a_{n_0} \left(\frac{L+1}{2}\right)^K$$

Thus the tail of $\sum a_n$ is bounded above by $a_{n_0}\left(\sum_{K=0}^{\infty} \left(\frac{L+1}{2}\right)^K\right)$

But $\sum_{K=0}^{\infty} \left(\frac{L+1}{2}\right)^K$ coverges (since $\frac{L+1}{2} < 1$).

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(ii) $r_n \to L > 1$, then for all large n, $r_n > \frac{L+1}{2} > 1$.
By argument analogous to (i) tail of $\sum a_n$ is bounded bellow by $a_{n_0}\left(\sum_{k=0}^{\infty}\left(\frac{L+1}{2}\right)^k\right)$ which diverges (since $\frac{L+1}{2} > 1$).

Thm: $\boxed{\text{The root test}}$ Let $\sum_{n=1}^{\infty} a_n$ be a series, & assume all $a_n > 0$; define a sequence $t_n := a_n^{\frac{1}{n}}$ (the sequence of termwise n^{th} roots), Then

(i) If t_n converges to a number strictly less than 1, then $\sum a_n$ converges.

(ii) If t_n converges to a number strictly greater than 1, then $\sum a_n$ diverges.

(iii) If t_n converges to 1, then the root test fails.

Proof: (i) If $t_n \to L < 1$ then for all n sufficiently large $(a_n)^{\frac{1}{n}} = t_n < \left(\frac{L+1}{2}\right) < 1$. Hence $a_n < \left(\frac{L+1}{2}\right)^n$; so the tail of the series is bounded above by a convergent geometric series.

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Examples: Let (a_n) be given $a_1=1$ $a_{n+1}=(\sin(\frac{1}{n}))a_n$.

Then $\sum_{n=1}^{\infty} a_n$ converges by ratio test:

$$\frac{a_{n+1}}{a_n}=\frac{(\sin(\frac{1}{n}))a_n}{a_n}=\sin(\tfrac{1}{n})\xrightarrow{n\to\infty}0<1$$

Let (b_n) be given $b_1=1$ $b_{n+1}=(1+\frac{1}{n})^n b_n$

Then $\sum_{n=1}^{\infty} b_n$ diverges by ratio test

$$\frac{b_{n+1}}{b_n}=(1+\tfrac{1}{n})^n\longrightarrow e>1$$

Ex Let (c_n) be given $c_1=1$ $c_{n+1}=\left(\sum_{m=1}^{n}(\frac{1}{3})^m\right)\cdot c_n$

$\sum c_n$ converge?

Ex does $\sum a_n$ converge when

(1) $a_1=1$ $a_{n+1}=\frac{3n^2+2n}{4n^2-n}a_n$

(2) $a_1=1$ $a_{n+1}=\frac{9n^3-n^2-n}{(2n+1)(2n+3)(2n+3)}a_n$

Ex $\sum_{n=1}^{\infty}\frac{7^n}{n^{n/2}}$

① Root test $\left(\frac{7^n}{n^{n/2}}\right)^{\frac{1}{n}}=\frac{7}{n^{\frac{1}{2}}}\xrightarrow{n\to\infty}0$ converges

② Ratio test $\frac{a_{n+1}}{a_n}=\frac{\frac{7^{n+1}}{(n+1)^{(n+1)/2}}}{\frac{7^n}{n^{n/2}}}=7\cdot\underbrace{\left(\frac{n}{n+1}\right)^{n/2}}_{\text{bounded}}\cdot\frac{1}{(n+1)^{\frac{1}{2}}}\Longrightarrow 0$

goes to zero thus

③ Comparison: Note $\frac{7^n}{n^{n/2}}=\left(\frac{7}{\sqrt{n}}\right)^n$ Hence for all n s.t. $\sqrt{n}\geq 8$ $\left(\frac{7^n}{n^{n/2}}\right)\leq\left(\frac{7}{8}\right)^n$. Hence tail is bounded above by a convergent geometric series & thus is convergent

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Exam Review