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Prop: Let A & B be finite sets. Then $A \times B$ is finite and
$$\#(A\times B) = \#(A)\cdot\#(B)$$
Proof: Let $A=\{a_1,\dots,a_r\}$ $B=\{b_1,\dots,b_t\}$, list and count the elements of $A\times B$.

Theorem: Let A & B be arbitrary sets, then
At least one of the sets A or B infinite $\Leftrightarrow A\cup B$ is infinite.
Proof: (Contradiction)

Examples

① $A=\{0\}$, $B=\mathbb{N}$ $\quad A\cup B=\mathbb{Z}_{\geq 0}$

② Suppose $A\cup B=\mathbb{Z}$ & A is finite subset of $\mathbb{Z}$, then B MUST be infinite.

Set identities (Prove these!)

Let A, B & C be arbitrary sets

① $A=(A\setminus B)\cup(A\cap B)$

② $A\cup B\cup C=(A\setminus B)\cup(B\setminus C)\cup(C\setminus A)\cup(A\cap B)\cup(B\cap C)\cup(A\cap C)$
$\qquad =(B\setminus A)\cup(C\setminus B)\cup(A\setminus C)\cup(A\cap B)\cup(B\cap C)\cup(A\cap C)$

③ Suppose A,B,C are pairwise disjoint (i.e, $A\cap B=B\cap C=C\cap A=\emptyset$) then $A\cap B\cap C=\emptyset$. The converse, however, is FALSE.

④ If A,B,C are pairwise disjoint, then
$$\#(A\cup B\cup C)=\#(A)+\#(B)+\#(C)$$

Induction

LEMMA: Suppose that $A\subseteq\mathbb{Z}_{\geq 0}$ s.t. (1) $0\in A$ &
② $\forall n\in\mathbb{Z}_{\geq 0}\ (n\in A\Rightarrow(n+1)\in A)$

use well-ordering see 1/29 notes

Then $A=\mathbb{Z}_{\geq 0}$. Proof by contradiction (hint let b be the smallest element of $\mathbb{Z}_{\geq 0}\setminus A$).

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Principal of Mathematical induction

LET $P(n)$ be a statement which depends on $n \in \mathbb{Z}_{\geq 0}$. Suppose that

(1) $P(0)$ is true

(2) $\forall n \in \mathbb{Z}_{\geq 0} \, (P(n) \Rightarrow P(n+1))$

Then $P(n)$ is true $\forall n \in \mathbb{Z}_{\geq 0}$.

Proof hint. Consider the set $A_P := \{n \in \mathbb{Z}_{\geq 0} \mid P(n) \text{ is true}\}$ can we apply lemma above?

Corollary: LET $Q(n)$ depend on $n \in \mathbb{Z}_{\geq 0}$. LET $n_0 \in \mathbb{Z}_{\geq 0}$ be fixed.

Suppose that

(1) $Q(n_0)$ is true

(2) $\forall n \geq n_0 \, (Q(n) \Rightarrow Q(n+1))$

Then $Q(n)$ is true $\forall n \in \mathbb{Z}$ s.t. $n \geq n_0$.

Examples

(1) $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(2) $1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

(3) $\sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^2 = \frac{(n+1)(2n+1)}{6n^2}$

(4) $P(n) :=$ "The sum of the first n even numbers is $n(n+1)$"

(5) $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ (do this w/o induction using (4))

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$P(x) :=$ "$f(w)$ is continuous at x"

Consider $Q :=$ "f has a max & min on $[a,b]$"

(What is this in terms of quantifiers? Eg "f has a max on $[a,b]$" becomes "$\exists y \in [a,b]$ s.t. $\forall x \in [a,b]$ $f(x) \leq f(y)$")

The sentence "$f(w)$ is continuous on $[a,b]$" is given in terms of quantifiers by "$\forall x \in [a,b], P(x)$", so what about the following theorem from calculus?

Thm: If $f(w)$ is a continuous function on $[a,b]$, then f has a maximum & a minimum on $[a,b]$.

Does this translate to

"$\forall x \in [a,b] \, (P(x) \Rightarrow \exists y \in [a,b], \exists w \in [a,b] \;\; f(w) \leq f(x) \leq f(y)$" ?

If not, how are they different and why?

Ex: Let $P(x,y) :=$ "$x+y>0$". The statements

(i) $\forall x \in \mathbb{R} \; \exists y \in \mathbb{R}$ s.t. $P(x,y)$

(ii) $\exists y \in \mathbb{R} \; \forall x \in \mathbb{R}$ s.t. $P(x,y)$

These statements are <u>NOT</u> equivalent. (Proof?)

Finite/ Infinite sets

Def: A set S is "finite" if $S = \emptyset$ or if $\exists n \in \mathbb{N}$ s.t. S has exactly n elements; a set is called "infinite" if it is not finite. For a finite set S define the "cardinality" of S (denoted $\#(S)$) to be 0 if $S = \emptyset$ or else $\#(S) = n$ where $n \in \mathbb{N}$ is the number of elements in S.

Examples: The following are infinite sets

(i) {Prime numbers}, {odd numbers}, {even numbers}, {integers which are multiples of n} (where n is some non-zero integer), $\mathbb{Z}$, $\mathbb{Q}$, etc.

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Notice that $\{odds\} \cap \{primes\}$ is infinite, but $\{odds\} \cap \{evens\} = \emptyset$ is finite. In particular, the intersection of two infinite sets may or may not be infinite.

Claim: The union of two infinite sets is infinite. (Proof by contradiction)

Prop: If A & B are finite sets, then

① $A \cup B$ & $A \cap B$ are finite

② $\#(A \cup B) \leq \#(A) + \#(B)$

③ $\#(A \cup B) = \#(A) + \#(B) \iff A \cap B = \emptyset$.

Exercise: If A, B, C are finite sets, then $A \cup B \cup C$ is finite & $\#(A \cup B \cup C) \leq \#(A) + \#(B) + \#(C)$.

Note $A \cap B \cap C = \emptyset$ is NOT enough to deduce that $\#(A \cup B \cup C) = \#(A) + \#(B) + \#(C)$. What is the necessary condition?

Theorem: For any pair of finite sets A & B,

$$\#(A \cup B) + \#(A \cap B) = \#(A) + \#(B).$$

Proof Hint: Note $A \cup B = (A \setminus B) \cup B$

& $(A \setminus B) \cap (B) = \emptyset$

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What's wrong with the following argument?

LET $P(n)$ = "The total number of legs in a group of n horses is always odd"

Suppose $P(n)$ is true for some n, then

$$\#(\text{legs in a group of } n+1 \text{ horses}) = \#(\text{legs on a horse}) + \#(\text{legs in a group of } n \text{ horses})$$
$$= 4 + \text{odd number} = \text{odd number}$$

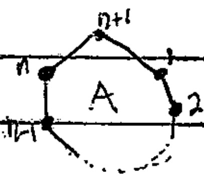
hence $P(n+1)$ is true. Thus $(P(n) \Rightarrow P(n+1))$ for any $n \in \mathbb{Z}_{\geq 0}$.

So $P(n)$ is true $\forall n \in \mathbb{Z}_{\geq 0}$

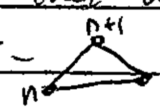
Prove: "For any convex n-gon A, the sum of the interior angles of A must be $\pi(n-2)$" is true $\forall n \geq 3$

$P(3)$ = sum of interior angles in a triangle is π ✓

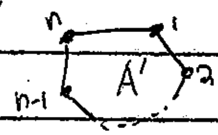
Suppose $P(n)$ is true. LET A be the $(n+1)$-gon drawn bellow



Draw a chord from n to 1, then we get A decomposed as a triangle T –



& a n-gon A'



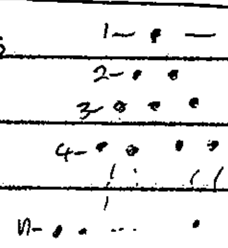
Note Sum angles of A = (Sum angles of T) + (Sum angles of A')
$$= \pi + \pi(n-2)$$
$$= \pi[(n+1)-2] \checkmark$$

Theorem: $\mathbb{Z}_{\geq 0}$ is well-ordered. That is to say, every non-empty subset $S \subseteq \mathbb{Z}_{\geq 0}$ has a minimal element.

→ Proof by contradiction noting for any $b \in S$ the set $\{x \in S \mid x \leq b\}$ is finite.

Triangular numbers

$T(n)$ total number of dots in a dotted triangle with n-rows



$T(1) = 1$
$T(2) = 1+2 = 3$
$T(3) = 1+2+3 = 6$
$T(n) = 1 + \cdots + n = \frac{n(n+1)}{2}$

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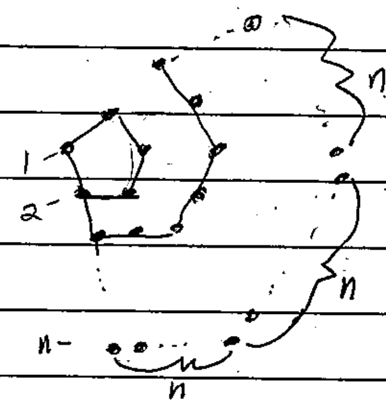
Pentagonal numbers

$P(n)$ = # of dots in a dotted pentagon with n-layers



Claim $P(n+1) = 4 + P(n) + 3(n-1) \quad \forall n \geq 1$

Thus

$$P(n) = P(n) - P(n-1) + P(n-1) - P(n-2) + P(n-2) - \cdots - P(1) + P(1)$$

$$= 4 + 3(n-2) + 4 + 3(n-3) + \cdots + 4 + 1$$

$$= 4n + 3\left(\frac{(n-1)n}{2}\right).$$