Appendix A
Some Elementary Topics

A.1 Sets

A *set* is a collection of objects. The objects in a set are called the *elements* (or *members*) of the set.

We might, for example, consider the set of capital letters appearing on this page, or the set of motorcycles licensed in Idaho, or the set of rational numbers. Suppose, however, that we wanted to find the set of rational people. Everybody might have a different collection. Which would be the right one? To avoid such problems we insist that sets be unambiguously defined. Collections based on highly subjective judgments—such as "all good football players" or "all likeable children"—are not sets.

Notation

To indicate that an object x is in the set A, we write

$$x \in A.$$

To indicate that x is not in A, we write

$$x \notin A.$$

Thus

$$\sqrt{2} \in \text{the set of real numbers} \quad \text{but} \quad \sqrt{2} \notin \text{the set of rational numbers}.$$

Sets are often denoted by braces. The set consisting of a alone is written $\{a\}$; that consisting of a, b is written $\{a, b\}$; that consisting of a, b, c, $\{a, b, c\}$; and so on. Thus

$$0 \in \{0, 1, 2\}, \quad 1 \in \{0, 1, 2\}, \quad 2 \in \{0, 1, 2\}, \quad \text{but} \quad 3 \notin \{0, 1, 2\}.$$

We can also use braces for infinite sets:

$\{1, 2, 3, \ldots\}$ is the set of positive integers,
$\{-1, -2, -3, \ldots\}$ is the set of negative integers,
$\{1, 2, 2^2, 2^3, \ldots\}$ is the set of powers of 2.

Sets are often defined by a property. We write $\{x: P\}$ to indicate *the set of all x for which property P holds*. Thus

$\{x: x > 2\}$ is the set of all numbers greater than 2;
$\{x: x^2 > 9\}$ is the set of all numbers whose squares are greater than 9;
$\{p/q: p, q \text{ integers}, q \neq 0\}$ is the set of all rational numbers.

If A is a set, then $\{x: x \in A\}$ is A itself.

Containment and Equality

If A and B are sets, then A is said to be *contained* in B, in symbols $A \subseteq B$, iff† every element of A is also an element of B. For example,

the set of equilateral triangles $\subseteq$ the set of all triangles,
the set of all college freshmen $\subseteq$ the set of all college students,
the set of rational numbers $\subseteq$ the set of real numbers.

If A is contained in B, then A is called a *subset* of B. Thus

the set of equilateral triangles is a subset of the set of all triangles,
the set of college freshmen is a subset of the set of all college students,
the set of rational numbers is a subset of the set of real numbers.

Two sets are said to be *equal* iff they have exactly the same membership. In symbols,

(A.1.1) $$\boxed{A = B \quad \text{iff} \quad A \subseteq B \text{ and } B \subseteq A.}$$

Examples

$$\{x: x^2 = 4\} = \{-2, 2\},$$
$$\{x: x^2 < 4\} = \{x: -2 < x < 2\},$$
$$\{x: x^2 > 4\} = \{x: x < -2 \text{ or } x > 2\}. \quad \square$$

The Intersection of Two Sets

The set of elements common to two sets A and B is called the *intersection* of A and

† By "iff" we mean "if and only if". This expression is used so often in mathematics that it is convenient to have an abbreviation for it.

B and is denoted by $A \cap B$. The idea is illustrated in Figure A.1.1. In symbols,

(A.1.2) $$\boxed{x \in A \cap B \quad \text{iff} \quad x \in A \quad \text{and} \quad x \in B.}$$

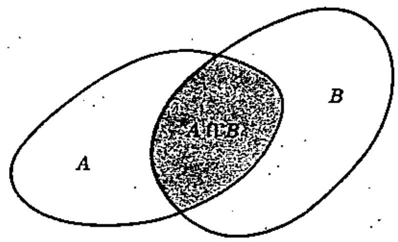


FIGURE A.1.1

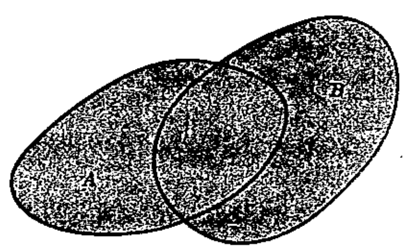
Examples

1. If A is the set of all nonnegative numbers, and B is the set of all nonpositive numbers, then $A \cap B = \{0\}$.
2. If A is the set of all multiples of 3, and B is the set of all multiples of 4, then $A \cap B =$ the set of all multiples of 12.
3. If $A = \{a, b, c, d, e\}$, and $B = \{c, d, e, f\}$, then $A \cap B = \{c, d, e\}$.
4. If $A = \{x: x > 1\}$ and $B = \{x: x < 4\}$, then $A \cap B = \{x: 1 < x < 4\}$. □

The Union of Two Sets

The *union* of two sets A and B, written $A \cup B$, is the set of elements which are either in A or in B. This does not exclude objects which are elements of both A and B. (See Figure A.1.2.) In symbols,

(A.1.3) $$\boxed{x \in A \cup B \quad \text{iff} \quad x \in A \quad \text{or} \quad x \in B.}$$



$A \cup B$
FIGURE A.1.2

Examples

1. If A is the set of all nonnegative numbers and B is the set of all nonpositive numbers, then $A \cup B =$ the set of all real numbers.
2. If $A = \{a, b, c, d, e\}$ and $B = \{c, d, e, f\}$, then $A \cup B = \{a, b, c, d, e, f\}$.

3. If $A = \{x: 0 < x < 1\}$ and $B = \{0, 1\}$, then $A \cup B = \{x: 0 \le x \le 1\}$.
4. If $A = \{x: x > 1\}$ and $B = \{x: x > 2\}$, then $A \cup B = \{x: x > 1\}$. □

The Empty Set

If the sets A and B have no elements in common, we say that A and B are *disjoint* and write $A \cap B = \emptyset$. We regard $\emptyset$ as a set with no elements and refer to it as the *empty set*.

Examples

1. If A is the set of all positive numbers, and B is the set of all negative numbers, then $A \cap B = \emptyset$.
2. If $A = \{0, 1, 2, 3\}$ and $B = \{4, 5, 6, 7, 8\}$, then $A \cap B = \emptyset$.
3. The set of all irrational rational numbers is empty; so is the set of all even odd integers; so is the set of real numbers whose squares are negative. □

The empty set $\emptyset$ plays a role in the theory of sets which is strikingly similar to the role played by 0 in the arithmetic of numbers. Without pursuing the matter very far, note that for numbers,

$$a + 0 = 0 + a = a, \qquad a \cdot 0 = 0 \cdot a = 0,$$

and for sets,

$$A \cup \emptyset = \emptyset \cup A = A, \qquad A \cap \emptyset = \emptyset \cap A = \emptyset.$$

Cartesian Products

If A and B are nonempty sets, then $A \times B$, the *Cartesian product* of A and B, is the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$. In set notation

(A.1.4) $$\boxed{A \times B = \{(a, b): a \in A,\ b \in B\}.}$$

Examples

1. For $A = \{0, 1\}$, $B = \{1, 2, 3\}$, $A \times B = \{(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3)\}$ and $B \times A = \{(1, 0), (1, 1), (2, 0), (2, 1), (3, 0), (3, 1)\}$.
2. If A is the set of rational numbers and B is the set of irrational numbers, then $A \times B$ is the set of all pairs (a, b) with a rational and b irrational. □

The Cartesian product $A \times B \times C$ consists of all ordered triples (a, b, c) with $a \in A$, $b \in B$, $c \in C$:

(A.1.5) $$\boxed{A \times B \times C = \{(a, b, c): a \in A, b \in B, c \in C.\}}$$

Exercises†

For Exercises 1–12 take

$$A = \{0, 2\}, \quad B = \{-1, 0, 1\}, \quad C = \{1, 2, 3, 4\},$$
$$D = \{2, 4, 6, 8, \ldots\}, \quad E = \{-2, -4, -6, -8, \ldots\},$$

and determine the following sets:

*1. $A \cup B$.	2. $A \cap B$.	*3. $B \cup C$.
*4. $B \cap C$.	5. $A \cup D$.	*6. $A \cap D$.
*7. $B \cap D$.	8. $D \cup E$.	*9. $C \cap D$.
*10. $A \times B$.	11. $B \times C$.	*12. $A \times B \times A$.

For Exercises 13–18 take

$$A = \{x: x > 2\}, \quad B = \{x: x \leq 4\}, \quad C = \{x: x > 3\},$$

and determine the following sets:

*13. $A \cup C$.	14. $A \cap C$.	*15. $B \cup C$.
*16. $B \cap C$.	17. $A \cap (B \cap C)$.	*18. $A \cap (B \cup C)$.

19. Given that $A \subseteq B$, find (a) $A \cup B$. (b) $A \cap B$.

*20. List all the nonempty subsets of $\{0, 1, 2, 3\}$.

21. What can you conclude about A and B given that
(a) $A \cup B = A$? (b) $A \cap B = A$? (c) $A \cup B = A$ and $A \cap B = A$?
Prove your answers.