Math 3283W
Spring 2010
Exam 3
04/23/10
Time Limit: 60 Minutes

Name (Print): KEY

Section and TA

This exam contains 7 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a theorem from either the lecture notes or the text, you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations and explanation will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; *clearly indicate* when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	20	
3	20	
4	15	
5	20	
Total:	100	

Be sure to answer a total of FIVE QUESTIONS: Check to see that you have all questions and pages. Answer the questions in the space provided on the question sheets. If you need extra space, write on the other side of the page, in this case please clearly indicate that your work is continued on the other side.

1. (25 points) For (a), (b), and (c), find the sum of the series. Show your work. (Note: this problem has 5 parts total).

(a) $\sum_{n=2}^{\infty} \frac{4^n-3^n}{5^n}$

$$\sum_{n=2}^{\infty} \frac{4^n-3^n}{5^n} = \sum_{n=2}^{\infty} \frac{4^n}{5^n} - \sum_{n=2} \frac{3^n}{5^n} = \sum_{n=2}^{\infty}\left(\frac{4}{5}\right)^n - \sum_{n=2}^{\infty}\left(\frac{3}{5}\right)^n$$

$$= \frac{\left(\frac{4}{5}\right)^2}{1-\frac{4}{5}} - \left[\frac{\left(\frac{3}{5}\right)^2}{1-\frac{3}{5}}\right] = \frac{16}{25-4} - \frac{9}{25-3} = \frac{16}{21} - \frac{9}{22}$$

(b) $\sum_{k=0}^{\infty} \frac{1}{2(k+3)(k+4)}$

$$\sum_{k=0}^{\infty} \frac{1}{2}\frac{1}{(k+3)(k+4)} = \frac{1}{2}\sum_{j=3}^{\infty}\frac{1}{j(j+1)} \quad \left[\text{REPLACE } k+3 \text{ BY } j\right]$$

$$= \frac{1}{2}\left(\sum_{j=1}^{\infty}\frac{1}{j(j+1)} - \underset{j=1}{\frac{1}{2}} - \underset{j=2}{\frac{1}{6}}\right) = \frac{1}{2}\left(1-\frac{2}{3}\right) = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}$$

(c) $\sum_{j=0}^{\infty}(-1)^{3j}(\frac{5}{6})^{3j}$

$$\sum_{j=0}^{\infty}(-1)^{3j}\left(\frac{5}{6}\right)^{3j} = \sum_{j=0}^{\infty}\left(-1\cdot\frac{5}{6}\right)^{3j} = \sum_{j=0}^{\infty}\left[-\left(\frac{5}{6}\right)^3\right]^j,$$

$$= \frac{1}{1-\left(-\frac{5}{6}\right)^3} \quad \left[\text{SINCE SERIES IS GEOMETRIC}\right]$$

$$= \frac{1}{1+\left(\frac{5}{6}\right)^3} = \frac{1}{1+\frac{125}{186}} = \frac{186}{186+125} = \frac{186}{311}$$

For (d) and (e) determine if the series converge or diverge. Show all of your reasoning.

(d) $\sum\limits_{n=2}^{\infty} \frac{n^{1/3}}{n^{5/3}-n}$

SINCE $\frac{n^{1/3}}{n^{5/3}}$ "LOOKS LIKE" $\frac{1}{n^{4/3}}$ AND $\sum\limits_{n=2}^{\infty} \frac{1}{n^{4/3}}$

CONVERGE, USE LIMIT COMPARISON TEST

$$\frac{\frac{n^{1/3}}{n^{5/3}-n}}{\frac{1}{n^{4/3}}} = \frac{n^{5/3}}{n^{5/3}-n} = \frac{1}{1-\frac{n}{n^{5/3}}} = \frac{1}{1-\frac{1}{n^{2/3}}} \longrightarrow 1$$

CONVERGES BY LIMIT COMPARISON

(e) $\sum\limits_{n=1}^{\infty} \frac{n!2^n}{(2n)!(2n+1)}$

USE RATIO TEST

$$\frac{\frac{(n+1)!\,2^{n+1}}{[2(n+1)]!\,(2(n+1)+1)}}{\frac{n!\,2^n}{(2n)!\,(2n+1)}} = \frac{(n+1)!}{n!} \cdot \frac{2^{n+1}}{2^n} \frac{(2n)!(2n+1)}{(2n+2)!(2n+3)}$$

$$= (n+1) \cdot 2 \cdot \frac{2n+1}{(2n+1)(2n+2)(2n+3)} = \frac{2(n+1)}{(2n+2)(2n+3)}$$

$$= \frac{2n+2}{4n^2+10n+6} \longrightarrow 0, \text{ AS } n \to \infty$$

CONVERGES

2. (20 points) Suppose $\sum_{n=1}^{\infty} a_n$ is a series of positive terms and its sequence of partial sums $s_n = a_1 + a_2 + \ldots + a_n$ satisfies $s_n \leq 9 + \frac{n-1}{n}$, for $n \geq 1$.

(a) Prove that $\sum_{n=1}^{\infty} a_n$ converges. Carefully show all of your reasoning.

(15p)

For all $n \geq 1$, $9 + \frac{n-1}{n} < 9 + 1 = 10$. Thus $\langle s_n \rangle$ is bounded above by 10. Moreover, since $a_n > 0$ $\forall n$, the sequence $\langle s_n \rangle$ is increasing. By the monotone convergence theorem, $\langle s_n \rangle$ converges; therefore $\sum_{n=1}^{\infty} a_n$ converges, since (by definition) $\lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} a_n$.

(b) What is the largest possible value for $\sum_{n=1}^{\infty} a_n$?

(5p)

The largest possible value is 10 (e.g. if $s_n = 9 + \frac{n-1}{n}$),

because $s_n \leq 9 + \frac{n-1}{n}$ $\forall n$

$$\Downarrow$$

$$\lim s_n \leq \lim \left(9 + \frac{n-1}{n}\right) = 10.$$

$$\parallel$$

$$\sum a_n$$

3. (20 points) Let $\sum_{n=1}^{\infty} a_n$, and $\sum_{n=1}^{\infty} b_n$ be series of positive terms such that for every $n \in \mathbb{N}$, $a_n \leq (\frac{n+10}{2n-1})b_n$.

(a) Does it follow that if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges? Give complete answers for your reasoning.

① $\frac{n+10}{2n-1} \leq 11$, SINCE $n+10 \leq 22n-11$ OR $21 \leq 21n$ FOR ALL n

SO $a_n \leq (\frac{n+10}{2n-1})b_n \leq 11 b_n$ FOR ALL n. $\sum b_n$ CONVERGES IMPL. $\sum 11 b_n$ CONVERGES. COMPARISON TEST $\Rightarrow$ $\sum a_n$ CONVERGES

② SINCE $\frac{(\frac{n+10}{2n-1})b_n}{b_n} = (\frac{n+10}{2n-1}) = \frac{1+\frac{10}{n}}{2-\frac{1}{n}} \rightarrow \frac{1}{2}$ AND $\sum b_n$ CONVERGES $\sum (\frac{n+10}{2n-1})b_n$ CONVERGES (LIMIT COMP). HENCE $\sum a_n$ CONVERGES: COMP TEST

③ SINCE $\frac{n+10}{2n-1} \rightarrow \frac{1}{2}$, $a_n \leq 1 \cdot b_n$ FOR n LARGE (ACT, $n \geq 11$). SO $\sum_{n=11}^{\infty} a_n$ AND TH[E] $\sum a_n$ CONVERGE

(b) What can you conclude about $\sum_{n=1}^{\infty} a_n$ if $\sum_{n=1}^{\infty} b_n$ diverges? Carefully describe your reasoning.

IF $a_n \leq \frac{n+10}{2n-1} b_n$ AND $\sum b_n$ DIVERGES, THEN

$\frac{1}{2} b_n \leq \frac{n+10}{2n-1} b_n$ FOR ALL n $\Rightarrow$ $\sum \frac{n+10}{2n-1} b_n$ DIVERGES

WE CAN NOT DETERMINE IF $\sum a_n$ CONVERGES OR DIVERGES IF TERMS ARE LESS THAN TERMS OF A DIVERGENT SERIES.

EXAMPLES $b_n \equiv 1$, $a_n = \frac{1}{n^2}$ CONVERGES

$b_n \equiv 1$, $a_n = \frac{n+10}{2n-1}$ DIVERGES

4. (15 points) Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive terms. Determine if the following series converge or diverge. Give reasons, but not a formal proof, for your answers.

(a) $\sum_{n=1}^{\infty} \ln(1+a_n)$

$\sum_1^{\infty} \ln(1+a_n)$ DOES CONVERGE

$\boxed{\text{L.C.T.}}$ We compute: (Noting that since $\sum_1^{\infty} a_n$ converges, $a_n \to 0$)

$$\lim_{n\to\infty} \frac{\ln(1+a_n)}{a_n} = \lim_{x\to 0} \frac{\ln(1+x)}{x} \underset{\text{L'Hop}}{=} \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1$$

Hence, by LCT, since $\sum_1^{\infty} a_n$ converges, so too must $\sum_1^{\infty} \ln(1+a_n)$

(b) $\sum_{n=1}^{\infty} (a_n)^3$

$\sum_1^{\infty} (a_n)^3$ DOES CONVERGE

$\boxed{\text{L.C.T.}}$ $$\lim_{n\to\infty} \frac{(a_n)^3}{a_n} = \lim_{n\to\infty} (a_n)^2 = \left(\lim_{n\to\infty} a_n\right)^2 = 0$$

since $a_n \to 0$ (noted above)

Since squaring is continuous

(c) $\sum_{n=1}^{\infty} (a_n)^{\frac{2}{3}}$

$\sum_1^{\infty} (a_n)^{2/3}$ may converge or diverge. Consider the following examples

① If $a_n = \frac{1}{n^{3/2}}$, then $\sum_1^{\infty} a_n$ converges (p-test) but $\sum_1^{\infty} (a_n)^{2/3} = \sum_1^{\infty} \frac{1}{n}$ diverges

② If $a_n = \frac{1}{n^2}$ then $\sum_1^{\infty} a_n$ converges & so does $\sum_1^{\infty} (a_n)^{2/3} = \sum_1^{\infty} \frac{1}{n^{4/3}}$.

5. (20 points) Let $\sum_{n=1}^{\infty} a_n$ be a series where the partial sums s_n satisfy the recurrence relationship $s_1 = 1$, and $s_{n+1} = (-1)^n n^{-\frac{5}{4}} + s_n$, for $n \geq 1$.

(a) Does a_n converge to 0? Show your reasoning.

Notice that $a_1 = s_1 = 1$, & that it's always true (for any series) that $s_{n+1} = a_{n+1} + s_n$. Hence $a_{n+1} = (-1)^n n^{-5/4}$ for each $n \geq 1$. The sequence $\frac{(-1)^n}{n^{5/4}}$ does converge to zero. We have $0 \leq \left|\frac{(-1)^n}{n^{5/4}}\right| \leq \frac{1}{n}$ since for each $n \geq 1$, $n \leq n^{5/4}$. By the pinching theorem $|a_{n+1}| \to 0$, thus $a_{n+1} \to 0$.

(b) Determine if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent or conditionally convergent. Give reasons for your answers.

We have $\sum_{n=1}^{\infty} a_n = 1 + \sum_{n=2}^{\infty} (-1)^{n-1}(n-1)^{-5/4} = 1 + \sum_{n=1}^{\infty} (-1)^n n^{-5/4}$.

This series is absolutely convergent by the p-test since $5/4 > 1$.

(c) If $L = \sum_{n=1}^{\infty} a_n$, find an estimate for $|s_{9,999} - L|$. You need not simplify your answer.

Note that $\frac{1}{n^{5/4}}$ is monotonically decreasing since the function $f(x) = x^{5/4}$ is monotonically increasing for $x \in (0, \infty)$. Thus the alternating series test applies & by the second part we have

$$|s_{9,999} - L| < a_{10,000} = 9{,}999^{-5/4}$$