Homework 3 Solutions.

PART A

1. Pf: By definition $a_n \to 0$ if, & only if given any $\varepsilon > 0$
$\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|a_n - 0| < \varepsilon$.
Now $|a_n - 0| = |a_n| = ||a_n| - 0|$.
Thus $a_n \to 0$ if & only if given any $\varepsilon > 0$
$\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $||a_n| - 0| < \varepsilon$.
But this is true if & only if $|a_n| \to 0$ (by definition of convergence

Putting all this together we see
$a_n \to 0$ if, & only if $|a_n| \to 0$ as desired.

2. a). We know that $\forall n \in \mathbb{N}$ $\cos(n\pi) = (-1)^n$.
So $a_n = \dfrac{\cos(n\pi)}{1 + n^{3/2}} = \dfrac{(-1)^n}{1+n^{3/2}}$.

b) We have $\lim\limits_{n\to\infty} \dfrac{(-1)^n}{1+n^{3/2}} = 0$. Observe that
$\forall n \in \mathbb{N}$ $1 + n^{3/2} \geq n$, & so $\forall n \in \mathbb{N}$ $0 \leq \dfrac{1}{1+n^{3/2}} \leq \dfrac{1}{n}$.
Since $\frac{1}{n} \to 0$ we have that $\dfrac{1}{1+n^{3/2}} \to 0$ by the pinching theorem. By problem 1

$$\frac{1}{1+n^{3/2}} \to 0 \iff \frac{(-1)^n}{1+n^{3/2}} \to 0 \text{ since}$$

$$\left| \frac{(-1)^n}{1+n^{3/2}} \right| = \frac{1}{1+n^{3/2}}.$$

3. We have $a_n = \sin\left(\frac{\pi n - 1}{1-2n} \ln\left(e \frac{(n^2-1)}{(n+1)^2}\right)\right)$

$$= \sin\left(\frac{\pi n - 1}{1-2n} \ln\left(e \frac{\cancel{(n+1)}(n-1)}{\cancel{(n+1)}(n+1)}\right)\right)$$

$$= \sin\left(\frac{\pi n - 1}{1-2n} \ln\left(e\left(\frac{n-1}{n+1}\right)\right)\right).$$

Notice that

$$\frac{\pi n - 1}{1-2n} = \frac{\pi - \frac{1}{n}}{\frac{1}{n} - 2} \longrightarrow -\frac{\pi}{2}$$

& $e\left(\frac{n-1}{n+1}\right) \longrightarrow e.$

By the continuity of $\ln(x)$ at $x = e$, we have that $\ln\left(e\left(\frac{n-1}{n+1}\right)\right) \to \ln(e) = 1.$

By the limit laws, since $\frac{\pi n - 1}{1-2n} \to -\pi/2$ & $\ln\left(e\left(\frac{n-1}{n+1}\right)\right) \to 1$ we have that

$$\left(\frac{\pi n - 1}{1-2n}\right)\left(\ln\left(e\left(\frac{n-1}{n+1}\right)\right)\right) \longrightarrow \left(-\frac{\pi}{2}\right)(1) = -\frac{\pi}{2}.$$

Finally by the continuity of the sine function we have that

$$\sin\left(\frac{\pi n - 1}{1-2n} \ln\left(e\left(\frac{n-1}{n+1}\right)\right)\right) \longrightarrow \sin\left(-\frac{\pi}{2}\right) = -1.$$

4. a) Proof by induction:

Notice that:

$$n=1 \Rightarrow \frac{3^n}{n!} = \frac{3}{1} \leq \frac{9}{2} \cdot \frac{3}{1}$$

$$n=2 \Rightarrow \frac{3^n}{n!} = \frac{3 \cdot 3}{1 \cdot 2} = \frac{9}{2} \leq \frac{9}{2} \cdot \frac{3}{2}$$

$$n=3 \Rightarrow \frac{3^n}{n!} = \frac{3 \cdot 3 \cdot 3}{1 \cdot 2 \cdot 3} = \frac{9}{2} \leq \frac{9}{2} \frac{3}{3}.$$

Now assume that $\frac{3^n}{n!} \leq \frac{9}{2} \cdot \frac{3}{n}$ for some $n \geq 3$.

Then we want to show that $\frac{3^{n+1}}{(n+1)!} \leq \frac{9}{2} \cdot \frac{3}{n+1}$.

We have

$$\frac{3^{n+1}}{(n+1)!} = \frac{3 \cdot 3^n}{(n+1)\, n!} \leq \frac{3}{(n+1)} \cdot \frac{9}{2} \cdot \frac{3}{n}$$

$$= \frac{9}{2} \cdot \frac{3}{(n+1)} \cdot \frac{3}{n}$$

$$\leq \frac{9}{2} \cdot \frac{3}{n+1}.$$

By induction the result holds.

b) We have that $\forall n \in \mathbb{N}$ $0 \leq a_n \leq \frac{9}{2} \frac{3}{n}$.

Since $\frac{9}{2} \cdot \frac{3}{n} \to 0$ it follows that $a_n \to 0$ by the pinching theorem.

PART B

5. a) Claim: $a_n \to \frac{1}{2}$.

Pf of claim: Let $\varepsilon > 0$. We have

$$\left| \frac{n^2-1}{2n^2+1} - \frac{1}{2} \right| = \left| \frac{2n^2-2 - (2n^2+1)}{2(2n^2+1)} \right|$$

$$= \left| \frac{2n^2-2-2n^2-1}{2(2n^2+1)} \right|$$

$$= \left| \frac{3}{2} \cdot \frac{1}{2n^2+1} \right|.$$

Thus

$$\left| \frac{n^2-1}{2n^2+1} - \frac{1}{2} \right| < \varepsilon \iff \frac{3}{2} \cdot \frac{1}{2n^2+1} < \varepsilon$$

$$\iff \frac{1}{2n^2+1} < \frac{2\varepsilon}{3}$$

$$\iff 2n^2+1 > \frac{3}{2\varepsilon}$$

$$\iff 2n^2 > \frac{3}{2\varepsilon} - 1$$

$$\iff n^2 > \frac{3}{4\varepsilon} - \frac{1}{2}.$$

Given $\varepsilon > 0$ we have $\forall\, n \geq \sqrt{\frac{3}{4\varepsilon} - \frac{1}{2}}$, $\left| a_n - \frac{1}{2} \right| < \varepsilon$.

By definition $\lim\limits_{n \to \infty} a_n = \frac{1}{2}$.

5 b). By part a)

$$\left|a_n - \frac{1}{2}\right| < \frac{1}{10^2} \quad \text{whenever}$$

$$n \geq \sqrt{\frac{3}{4\left(\frac{1}{100}\right)} - \frac{1}{2}} = \sqrt{\frac{300}{4} - \frac{1}{2}}$$

$$= \sqrt{\frac{298}{4}}$$

$$= \sqrt{\frac{149}{2}}.$$

6.

a) $\langle a_n + b_n \rangle$ is divergent $\Rightarrow$ $\langle a_n \rangle$ is divergent or $\langle b_n \rangle$ is divergen

Ans: True. Proof: Consider the contrapositive:

$\langle a_n \rangle$ & $\langle b_n \rangle$ are convergent $\Rightarrow$ $\langle a_n + b_n \rangle$ is convergent.

This follows immediately from theorem 2.1 a).

b) $\langle a_n \rangle$ & $\langle a_n + b_n \rangle$ are convergent $\Rightarrow$ $\langle b_n \rangle$ is convergent.

Ans: True. Proof: By theorem 2.1 b) $\langle a_n \rangle$ is convergent

$\Rightarrow$ $\langle -a_n \rangle$ is convergent.

By theorem 2.1 a), $\langle -a_n \rangle$ & $\langle a_n + b_n \rangle$ are convergent

$\Rightarrow$ $\langle (a_n + b_n) + (-a_n) \rangle = \langle b_n \rangle$ is convergent.

c) $\langle a_n \rangle$ & $\langle a_n b_n \rangle$ are convergent $\Rightarrow$ $\langle b_n \rangle$ is convergent.

Ans: False. Counter-example: Let $a_n = 0$ $\forall n \in \mathbb{N}$,

& let $b_n = 10^{10^{10^{10^{n}}}}$ $\forall n \in \mathbb{N}$.

Then $\langle a_n \rangle = \langle 0 \rangle = \langle a_n b_n \rangle$ is convergent, but hopefully it is clear that $\langle b_n \rangle$ is divergent.

7. We are given $s_n = \sum_{i=0}^{n} \frac{1}{2^i}$.

Recall that when $x \neq 1$ we have

$$\sum_{j=0}^{n} x^n = \frac{1-x^{n+1}}{1-x},$$

since $(1+x+x^2+\cdots+x^n)(1-x) = 1-x^{n+1}$.

Thus $s_n = \sum_{i=0}^{n} \frac{1}{2^i} = \sum_{i=0}^{n} \left(\frac{1}{2}\right)^i = \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = 2\left(1-\frac{1}{2^{n+1}}\right)$

$= 2 - \frac{1}{2^n}$.

Now $\frac{1}{2^n} \to 0$ since for example $\frac{1}{k} \to 0$ & $\langle \frac{1}{2^n} \rangle$ is a subsequence of $\langle \frac{1}{k} \rangle$.

Given $\varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\forall m \geq N$ $\frac{1}{2^m} < \varepsilon$,

hence $\forall m \geq N$ $\left|\left(2-\frac{1}{2^m}\right)-2\right| = \left|\frac{1}{2^m}\right| < \varepsilon$.

Therefore $\lim_{n\to\infty} s_n = 2$.