MATH 3283W HOMEWORK # 5 DUE 4/6/10

SUMMATION NOTATION, SECTIONS 4.2, 4.3, CHAPTER 4

PART A

1. DETERMINE IF THE FOLLOWING SERIES CONVERGE OR DIVERGE. GIVE REASONS FOR YOUR ANSWERS.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{5^{3n+2}}{4^{1+4n}}$ (c) $\displaystyle\sum_{m=0}^{\infty} \frac{1}{(m+1)(m+2)}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ d) $\displaystyle\sum_{n=1}^{\infty} n^{-5/6}$

2 (a) FIND A FORMULA FOR $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} ij$ IN TERMS OF n. SHOW YOUR WORK

(b) EVALUATE $\displaystyle\sum_{i=1}^{6}\sum_{j=1}^{6} ij$

3 FIND THE SUM OF THE FOLLOWING SERIES. FOR WHAT VALUES OF X DOES IT CONVERGE?

(a) $\displaystyle\sum_{n=2}^{\infty} x^{3n}$ (b) $\displaystyle\sum_{n=K}^{\infty} (-1)^n x^{2n}$, WHERE $K \in \mathbb{N}$

4 (a) WRITE THE FOLLOWING INIFINITE DECIMALS AS FRACTIONS

(i) $.\overline{357}$ (ii) 123211232112321......

(b) WHAT IS THE VALUE OF THE SUM

$.\overline{59} + .\overline{73}$?

PART B

5, SUPPOSE $\displaystyle\sum_{n=1}^{\infty} a_n$ IS A SERIES OF POSITIVE TERMS WHICH CONVERGES. DO THE FOLLOWING SERIES CONVERGE OR DIVERGE? PROVE YOUR ANSWERS

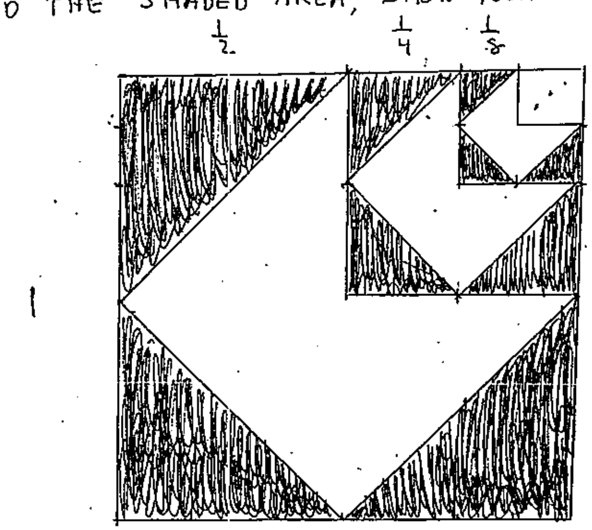
(a) $\displaystyle\sum_{n=1}^{\infty} \sin a_n$ (b) $\displaystyle\sum_{n=1}^{\infty} \cos a_n$

6 DETERMINE IF THE SERIES CONVERGE OR DIVERGE. GIVE CAREFUL REASONS FOR YOUR ANSWERS

(a) $1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^2}+\frac{1}{5^3}+\cdots$ (b) $1+\frac{1}{2^3}+\frac{1}{3}+\frac{1}{4^3}+\frac{1}{5}+\cdots$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{(1+1/n)}}$

7 FIND THE SHADED AREA, SHOW YOUR REASONING/CALCULATIONS



Homework #5 Solutions

PART A

1. a) We have $\sum_{n=1}^{\infty} \frac{5^{3n+2}}{4^{1+4n}} = \frac{25}{4} \sum_{n=1}^{\infty} \left(\frac{125}{256}\right)^n$.

This series converges since it is a geometric series & $0 < \frac{125}{256} < 1$.

b) Observe that $\forall\, n \geq 1$, $n^2+n > n^2$ & so $\frac{1}{n^2+n} < \frac{1}{n^2}$.

We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (by the p-test) thus $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is also convergent by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

c) We have $\sum_{m=0}^{\infty} \frac{1}{(m+1)(m+2)} = \frac{1}{2} + \sum_{m=1}^{\infty} \frac{1}{(m+1)(m+2)} = \frac{1}{2} + \sum_{m=1}^{\infty} \frac{1}{m^2+3m+2}$

As in (b) we have $\forall\, m \geq 1$, $m^2+3m+2 > m^2$ & so $\frac{1}{m^2+3m+2} < \frac{1}{m^2}$. Again, by comparison with $\sum_{m=1}^{\infty} \frac{1}{m^2}$, we see that $\sum_{m=0}^{\infty} \frac{1}{(m+1)(m+2)}$ is convergent.

d) The series $\sum_{n=1}^{\infty} n^{-5/6} = \sum_{n=1}^{\infty} \frac{1}{n^{5/6}}$ is divergent by the p-test.

2. a) We know that $1+2\cdots+n = \frac{n(n+1)}{2}$.

Thus $\sum_{i=1}^{n}\sum_{j=1}^{n} i\cdot j = \sum_{i=1}^{n} i \sum_{j=1}^{n} j = \sum_{i=1}^{n} i\,\frac{n(n+1)}{2}$

$$= \frac{n(n+1)}{2}\sum_{i=1}^{n} i$$

$$= \left(\frac{n(n+1)}{2}\right)^2$$

b). Using the derived formula from part a)

we have

$$\sum_{i=1}^{6}\sum_{j=1}^{6} i\cdot j = \left(\frac{6(6+1)}{2}\right)^2 = \left(\frac{42}{2}\right)^2 = (21)^2 = 441.$$

3. a) We have

$$\sum_{n=2}^{\infty} x^{3n} = \sum_{n=2}^{\infty} (x^3)^n.$$

We know that this geometric series is convergent iff $|x^3|<1$, i.e. $|x|<1$. If $|x|<1$ then

$$\sum_{n=2}^{\infty} (x^3)^n = \sum_{n=0}^{\infty} (x^3)^n - 1 - x^3 = \frac{1}{1-x^3} - 1 - x^3$$

$$= \frac{1-1+x^3-x^3+x^6}{1-x^3} = \frac{x^6}{1-x^3}$$

3b) We have

$$\sum_{n=k}^{\infty}(-1)^n x^{2n} = \sum_{n=0}^{\infty}(-x^2)^n - \sum_{n=0}^{k-1}(-x^2)^n .$$

We know this geometric series converges iff $|-x^2| < 1$, i.e. $|x|<1$. If $|x|<1$ we can apply the known formulae for a geometric series & a finite geometric sum to get

$$\sum_{n=k}^{\infty}(-1)^n x^{2n} = \frac{1}{1-(-x^2)} - \frac{1-(-x^2)^k}{1-(-x^2)}$$

$$= \frac{1-1+(-1)^k x^{2k}}{1+x^2} = \frac{(-1)^k x^{2k}}{1+x^2} .$$

4a) i) $.\overline{357} = \sum_{n=1}^{\infty} \frac{357}{10^{3n}} = 357\left(\sum_{n=0}^{\infty}\left(\frac{1}{1000}\right)^n - 1\right)$

$$= 357\left(\frac{1}{1-\frac{1}{1000}} - 1\right)$$

$$= 357\left(\frac{1}{999}\right) = \frac{357}{999} = \frac{119}{333}$$

ii) $.12321\,12321\,\overline{12321} = \sum_{n=1}^{\infty} \frac{12321}{10^{5n}} = 12321\left(\sum_{n=1}^{\infty}\left(\frac{1}{100000}\right)^n\right)$

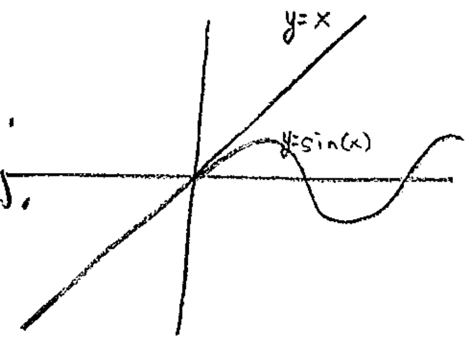
$$= \frac{12321}{99999} = \frac{4107}{33333} = \frac{1369}{11111}$$

4b) $.\overline{59} + .\overline{73} = \sum_{n=1}^{\infty} \frac{59}{100^n} + \sum_{n=1}^{\infty} \frac{73}{100^n}$

(since both series converge) $= (59+73) \sum_{n=1}^{\infty} \left(\frac{1}{100}\right)^n$

$= \frac{132}{99} = \frac{4}{3}$

PART B



5. a) Note: $\forall x \in \mathbb{R}$, $x \geq 0$ we have $|\sin(x)| \leq x$.

Thus $0 \leq |\sin(a_n)| \leq a_n$, for each $n \in \mathbb{N}$.
Since $\sum_{n=1}^{\infty} a_n$ is convergent, the comparison test tells us that $\sum_{n=1}^{\infty} |\sin(a_n)|$ is <u>convergent</u>. It follows that $\sum_{n=1}^{\infty} \sin(a_n)$ is convergent since it is absolutely convergent.

b) Since $\cos(x)$ is a continuous function we have $\lim_{n\to\infty} \cos(a_n) = \cos\left(\lim_{n\to\infty} a_n\right)$. Since $\sum a_n$ is convergent we know $a_n \to 0$. Thus $\lim_{n\to\infty} \cos(a_n) = \cos(0) = 1$. Since $\cos(a_n) \not\to 0$ we have that $\sum^{\infty} \cos(a_n)$ must be <u>divergent</u>.

6. a) Note $1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^2} + \frac{1}{5^3} + \cdots + \frac{1}{(2n-1)^3} + \frac{1}{(2n)^2}$

$$= \sum_{k=1}^{n} \left(\frac{1}{(2n-1)^3} + \frac{1}{(2n)^2} \right)$$

$$= \sum_{k=1}^{n} \frac{1}{(2k-1)^3} + \sum_{k=1}^{n} \frac{1}{(2k)^2}$$

Now $\frac{1}{(2n-1)^3} < \frac{1}{n^3}$ & $\frac{1}{(2n)^2} < \frac{1}{n^2}$ for each n.

By the p-test $\sum_{n=1}^{\infty} \frac{1}{k^3}$ & $\sum_{k=1}^{\infty} \frac{1}{k^2}$ are convergent.

thus $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^3}$ & $\sum_{k=1}^{\infty} \frac{1}{(2k)^2}$ are convergent by the comparison theorem. Since both of these series converge we have that their sum is convergent & $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^3} + \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \sum_{k=1}^{\infty} \left(\frac{1}{(2k-1)^3} + \frac{1}{(2k)^2} \right)$.

b) Let $a_n = \frac{1}{2n-1}$ & $b_n = \frac{1}{2n-1} + \frac{1}{(2n)^3}$ for each $n \in \mathbb{N}$.

It is clear that $0 \le a_n \le b_n$ for each $n \in \mathbb{N}$. Thus, by the comparison theorem, if $\sum_{n=1}^{\infty} a_n$ is divergent $\sum_{n=1}^{\infty} b_n$ is divergent. We should note that

$$1 + \frac{1}{2^3} + \frac{1}{3} + \frac{1}{4^3} + \frac{1}{5} + \cdots = \sum_{n=1}^{\infty} b_n.$$

Notice that $\lim_{n \to \infty} \frac{1/n}{1/(2n-1)} = \lim_{n \to \infty} \frac{2n-1}{n} = 2 > 0$. Thus by the limit comparison test $\sum \frac{1}{2n-1}$ diverges since $\sum \frac{1}{n}$ diverges.

6c) Notice that by the continuity of $\ln(x)$ & e^x, we have

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} e^{\ln(n^{1/n})} = \lim_{n\to\infty} e^{\frac{1}{n}\ln(n)} = e^{\lim_{n\to\infty}\frac{\ln(n)}{n}}$$

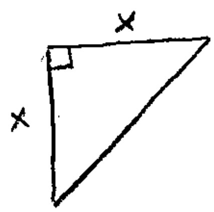
$$= e^0 = 1.$$

Also $\dfrac{1/n}{1/n^{1+1/n}} = \dfrac{n^{1+1/n}}{n} = n^{1/n}$.

Thus $\displaystyle\lim_{n\to\infty} \frac{1/n}{1/n^{1+\frac{1}{n}}} = 1$.

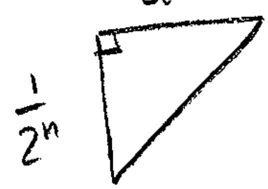
Since $\sum \frac{1}{n}$ is divergent we conclude that $\sum \frac{1}{n^{1+\frac{1}{n}}}$ is divergent by the limit comparison theorem.

7. Notice that the area of the triangle



is $\frac{x^2}{2}$. At the n^{th} step we add 3 triangles

with dimensions $\frac{1}{2^n}$



so the additional area is $3 \cdot \frac{1}{2^{2n+1}}$.

The area of the shaded area is thus

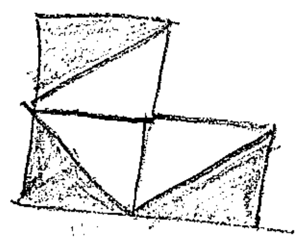
$$\sum_{n=1}^{\infty} 3 \cdot \frac{1}{2^{2n+1}} = \frac{3}{2} \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \frac{3}{2} \cdot \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n$$

$$= \frac{3}{8} \left(\frac{1}{1 - 1/4} \right)$$

$$= \frac{3}{8} \left(\frac{4}{3} \right) = \frac{1}{2}.$$

7 cont.

Note: A slicker way to do this problem is simply observing that at each step the amount of shaded area is exactly equal to the amount of unshaded area:



Thus the shaded region is half the area of the whole square.