ALMOST CONSERVATION LAWS AND GLOBAL ROUGH SOLUTIONS TO A NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT. We prove an "almost conservation law" to obtain global-in-time well-posedness for the cubic, defocussing nonlinear Schrödinger equation in $H^s(\mathbf{R}^n)$ when $n = 2, 3$ and $s > \frac{4}{7}, \frac{5}{6}$, respectively.

1. Introduction and Statement of Results

We study the following initial value problem for a defocussing nonlinear Schrödinger equation,

$$i\partial_t\phi(x,t) + \Delta\phi(x,t) = |\phi(x,t)|^2\phi(x,t) \quad x \in \mathbf{R}^n, t \geq 0 \tag{1.1}$$

$$\phi(x,0) = \phi_0(x) \ \in H^s(\mathbf{R}^n) \tag{1.2}$$

when $n = 2, 3$. Here $H^s(\mathbf{R}^n)$ denotes the usual inhomogeneous Sobolev space. Our goal is to loosen the regularity requirements on the initial data which ensure global-in-time solutions. In particular, we aim to extend the global theory to certain infinite energy initial data.

It is known [5] that (1.1)-(1.2) is well-posed locally in time when $n = 2, 3$ and $s > 0, \frac{1}{2}$ respectively[1]. In addition, these local solutions enjoy L^2 conservation;

$$||\phi(\cdot,t)||_{L^2(\mathbb{R}^n)} = ||\phi_0(\cdot)||_{L^2(\mathbb{R}^n)} \tag{1.3}$$

and the $H^1(\mathbf{R}^n)$ solutions have the following conserved energy,

$$E(\phi)(t) \equiv \int_{\mathbb{R}^n} \frac{1}{2}|\nabla_x\phi(x,t)|^2 + \frac{1}{4}|\phi(x,t)|^4 \ dx \quad = \ E(\phi)(0). \tag{1.4}$$

Together, energy conservation and the local-in-time theory immediately yield global-in-time well-posedness of (1.1)-(1.2) from data in $H^s(\mathbb{R}^n)$ when $s \geq 1$, and $n = 2, 3$. It is conjectured that (1.1)-(1.2) is in fact globally well-posed in time from all data included in the local theory. The obvious impediment to claiming global-in-time solutions in H^s, with $0 < s < 1$, is the lack of any applicable conservation law.

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[1]In addition, there are local in time solutions from $L^2, H^{\frac{1}{2}}$ data when $n = 2, 3$, respectively. However, it is not yet known whether the time interval of existence for such solutions depends only on the data's Sobolev norm. For example, the L^2 conservation law (1.3) does not yield the widely conjectured result of global in time solutions on $\mathbf{R}^{2+1}$ from L^2 initial data.

The first argument extending the lifespan of rough solutions to (1.1)-(1.2) in a range $s_0 < s < 1$ was given in [2] (see also [3]). In what might be called a "Fourier truncation" approach, Bourgain observed that from the point of view of regularity, the high frequency component of the solution ϕ is well-approximated by the corresponding linear evolution of the data's high frequency component. More specifically: one makes a first approximation to the solution for a small time step by evolving the high modes linearly, and the low modes according to the nonlinear flow for which one has energy conservation. The correction term one must add to match this approximation with the actual solution is shown to have finite energy. This correction is added to the low modes as data for the nonlinear evolution during the next time step, where the high modes are again evolved linearly. For $s > \frac{3}{5}$, one can repeat this procedure to an arbitrarily large time provided the distinction between "high" and "low" frequencies is made at sufficiently large frequencies.

The argument in [2] has been applied to other subcritical initial value problems with sufficient smoothing in their principal parts. (See [3], [7], [14], [19], [23], and [24]). It is important to note that the Fourier truncation method demonstrates more than just rough data global existence. Indeed, write S_t^{NL} for the nonlinear flow[2] of (1.1)-(1.2), and let S_t^L denote the corresponding *linear* flow. The Fourier truncation method shows then that for $s > \frac{3}{5}$ and for all $t \in [0, \infty)$,

$$S_t^{NL}\phi_0 - S_t^L\phi_0 \in H^1(\mathbf{R}^2). \tag{1.5}$$

Besides being part of the conclusion, the smoothing property (1.5) seems to be a crucial constituent of the Fourier truncation argument itself.

In this paper we will use a modification of the above arguments, originally put forward to analyze equations where the smoothing property (1.5) is not available because it is either false (e.g. Wave maps [17][3]) or simply not known (e.g. Maxwell-Klein-Gordon equations [16], for which we suspect (1.5) is false). In this "almost conservation law" approach, one controls the growth in time of a rough solution by monitoring the energy of a certain smoothed out version of the solution. It can be shown that the energy of the smoothed solution is "almost conserved" as time passes, and controls the solution's sub-energy Sobolev norm. In proving the almost conservation law for the i.v.p. (1.1)-(1.2), we shall use only the linear estimates presented in [2], [3]. Implicitly, we also use the view of [2] that the energy at high frequencies does not move rapidly to low frequencies.

The almost conservation approach to global rough solutions has proven to be quite robust [17], [16], [9], [12], and has been improved significantly by adding additional "correction" terms to the original almost conserved energy functional. As a result, one obtains even stronger bounds on the growth of the solution's rough norm, and at least in some cases sharp global well-posedness results [13], [10], [11].

The aims of this paper are three-fold: first and most obviously, an improved understanding of the evolution properties of rough solutions of (1.1)-(1.2); second, the almost conservation law approach is presented in a relatively straightforward setting; and third, we can directly compare this almost conservation law approach to the Fourier cut-off technique, since both approaches apply to the semilinear Schrödinger initial value problem. Our main result is the following:

Theorem 1.1. *The initial value problem* (1.1)-(1.2) *is globally-well-posed from data* $\phi_0 \in H^s(\mathbf{R}^n)$, $n = 2, 3$ *when* $s > \frac{4}{7}, \frac{5}{6}$ *respectively.*

[2] That is, $S_t^{NL}(\phi_0)(x) = \phi(x,t)$, where ϕ, ϕ_0 as in (1.1)-(1.2).

[3] See the appendix of [16] for the failure of (1.5) for Wave Maps.

By "globally-well-posed", we mean that given data $\phi_0 \in H^s(\mathbf{R}^n)$ as above, and any time $T > 0$, there is a unique solution to (1.1)-(1.2)

$$\phi(x,t) \in C([0,T]; H^s(\mathbf{R}^n)) \tag{1.6}$$

which depends continuously in (1.6) upon $\phi_0 \in H^s(\mathbf{R}^n)$. The polynomial bounds we obtain for the growth of $||\phi||_{H^s(\mathbf{R}^n)}(t)$ are contained in (3.4), (3.14), and (4.6) below.

Theorem 1.1 extends to some extent the work in [2, 3] where global well-posedness was shown when $s > \frac{3}{5}, \frac{11}{13}$ and $n = 2, 3$ respectively. In a different sense, the result here is weaker than the results of [2, 3] as we obtain no information whatsoever along the lines of (1.5).

In a later paper, we hope to extend Theorem 1.1 to still rougher data, using the additional cancellation terms mentioned above, and the multilinear estimates contained in [6].

In Section 2 below we present some notation and linear estimates that are used in our proofs. Sections 3, 4 present the almost conservation laws and proofs of Theorem 1.1 in space dimensions two and three, respectively.

2. Estimates, Norms, and Notation

Given $A, B \geq 0$, we write $A \lesssim B$ to mean that for some universal constant $K > 2$, $A \leq K \cdot B$. We write $A \sim B$ when both $A \lesssim B$ and $B \lesssim A$. The notation $A \ll B$ denotes $B > K \cdot A$.

We write $\langle A \rangle \equiv (1 + A^2)^{\frac{1}{2}}$, and $\langle \nabla \rangle$ for the operator with Fourier multiplier $(1 + |\xi|^2)^{\frac{1}{2}}$. The symbol ∇ will denote the spatial gradient.

We will use the weighted Sobolev norms, (see [22, 1, 4, 21]),

$$||\psi||_{X_{s,b}} \equiv ||\langle \xi \rangle^s \langle \tau - |\xi|^2 \rangle^b \tilde{\psi}(\xi, \tau)||_{L^2(\mathbf{R}^n \times \mathbf{R})}. \tag{2.1}$$

Here $\tilde{\psi}$ is the space-time Fourier transform of ψ. We will need local-in-time estimates in terms of truncated versions of the norms (2.1),

$$||f||_{X^\delta_{s,b}} \equiv \inf_{\psi = f \text{on} [0,\delta]} ||\psi||_{X^\delta_{s,b}}. \tag{2.2}$$

We will often use the notation $\frac{1}{2}+ \equiv \frac{1}{2} + \epsilon$ for some universal $0 < \epsilon \ll 1$. Similarly, we shall write $\frac{1}{2}- \equiv \frac{1}{2} - \epsilon$, and $\frac{1}{2} - - \equiv \frac{1}{2} - 2\epsilon$.

Given Lebesgue space exponents q, r and a function $F(x,t)$ on $\mathbf{R}^{n+1}$, we write

$$||F||_{L^q_t L^r_x(\mathbf{R}^{n+1})} \equiv \left(\int_{\mathbf{R}} \left(\int_{\mathbf{R}^n} |F(x,t)|^r dx \right)^{\frac{q}{r}} dt \right)^{\frac{1}{q}}. \tag{2.3}$$

This norm will be shortened to $L^q_t L^r_x$ for readability, or to $L^r_{x,t}$ when $q = r$.

We will need Strichartz-type estimates [25, 15, 18] involving the spaces (2.3), (2.1). We will call a pair of exponents (q, r) *Schrödinger admissible* for $\mathbf{R}^{n+1}$ when $q, r \geq 2$, $(n, q) \neq (2, 2)$, and

$$\frac{1}{q} + \frac{n}{2r} = \frac{n}{4}. \tag{2.4}$$

For a Schrödinger admissible pair (q, r) we have what we will call the $L^q_t L^r_x$ Strichartz estimate,

$$||\phi||_{L^q_t L^r_x(\mathbf{R}^{n+1})} \lesssim ||\phi||_{X_{0,\frac{1}{2}+}}. \tag{2.5}$$

Finally, we will need a refined version of these estimates due to Bourgain [2].

Lemma 2.1. *Let $\psi_1, \psi_2 \in X^\delta_{0,\frac{1}{2}+}$ be supported on spatial frequencies $|\xi| \sim N_1, N_2$, respectively. Then for $N_1 \leq N_2$, one has*

$$||\psi_1 \cdot \psi_2||_{L^2([0,\delta]\times\mathbf{R}^2)} \lesssim \left(\frac{N_1}{N_2}\right)^{\frac{1}{2}} ||\psi_1||_{X^\delta_{0,\frac{1}{2}+}} ||\psi_2||_{X^\delta_{0,\frac{1}{2}+}}. \tag{2.6}$$

In addition, (2.6) *holds (with the same proof) if we replace the product $\psi_1 \cdot \psi_2$ on the left with either $\overline{\psi}_1 \cdot \psi_2$ or $\psi_1 \cdot \overline{\psi}_2$.*

3. Almost conservation and Proof of Theorem 1.1 in $\mathbf{R}^2$

For rough initial data, (1.2) with $s < 1$, the energy is infinite, and so the conservation law (1.4) is meaningless. Instead, Theorem 1.1 rests on the fact that a smoothed version of the solution (1.1)-(1.2) has a finite energy which is *almost* conserved in time. We express this 'smoothed version' as follows.

Given $s < 1$ and a parameter $N \gg 1$, define the multiplier operator

$$\widehat{I_N f}(\xi) \equiv m_N(\xi)\hat{f}(\xi), \tag{3.1}$$

where the multiplier $m_N(\xi)$ is smooth, radially symmetric, nonincreasing in $|\xi|$ and

$$m_N(\xi) = \begin{cases} 1 & |\xi| \leq N \\ \left(\frac{N}{|\xi|}\right)^{1-s} & |\xi| \geq 2N. \end{cases} \tag{3.2}$$

For simplicity, we will eventually drop the N from the notation, writing I and m for (3.1) and (3.2). Note that for solution and initial data ϕ, ϕ_0 of (1.1), (1.2), the quantities $||\phi||_{H^s(\mathbb{R}^n)}(t)$ and $E(I_N\phi)(t)$ (see (1.4)) can be compared,

$$E(I_N\phi)(t) \leq \left(N^{1-s}||\phi(\cdot,t)||_{\dot{H}^s(\mathbb{R}^n)}\right)^2 + ||\phi(t,\cdot)||^4_{L^4(\mathbb{R}^n)}, \tag{3.3}$$

$$||\phi(\cdot,t)||^2_{H^s(\mathbb{R}^n)} \lesssim E(I_N\phi)(t) + ||\phi_0||^2_{L^2(\mathbb{R}^n)}. \tag{3.4}$$

Indeed, the $\dot{H}^1(\mathbb{R}^n)$ component of the left hand side of (3.3) is bounded by the right side by using the definition of I_N and by considering separately those frequencies $|\xi| \leq N$ and $|\xi| \geq N$. The L^4 component of the energy in (3.3) is bounded by the right hand side of (3.3) by using (for example) the Hörmander-Mikhlin multiplier theorem. The bound (3.4) follows quickly from (3.2) and L^2 conservation (1.3) by considering separately the $\dot{H}^s(\mathbf{R}^n)$ and $L^2(\mathbf{R}^n)$ components of the left hand side of (3.4).

To prove Theorem 1.1, we may assume that $\phi_0 \in C^\infty_0(\mathbf{R}^n)$, and show that the resulting global-in-time solution grows at most polynomially in the H^s norm,

$$||\phi(\cdot,t)||_{H^s(\mathbb{R}^n)} \leq C_1 t^M + C_2, \tag{3.5}$$

where the constants C_1, C_2, M depend only on $||\phi_0||_{H^s(\mathbf{R}^n)}$ and not on higher regularity norms of the smooth data. Theorem 1.1 follows immediately from (3.5), the local-in-time theory [5], and a standard density argument.

By (3.4), it suffices to show

$$E(I_N\phi)(t) \lesssim (1+t)^M. \tag{3.6}$$

for some $N = N(t)$. (See (3.13), (3.14) below for the definition of N and the growth rate M we eventually establish.) The following proposition, which is one of the two main estimates of this paper (see also Proposition 4.1), represents an "almost conservation law" of the title and will yield (3.6) in space dimension $n = 2$.

Proposition 3.1. *Given $s > \frac{4}{7}, N \gg 1$, and initial data $\phi_0 \in C_0^\infty(\mathbf{R}^2)$ (see preceeding remark) with $E(I_N\phi_0) \leq 1$, then there exists a $\delta = \delta(||\phi_0||_{L^2(\mathbf{R}^2)}) > 0$ so that the solution*

$$\phi(x,t) \in C([0,\delta], H^s(\mathbf{R}^2))$$

of (1.1)-(1.2) *satisfies*

$$E(I_N\phi)(t) = E(I_N\phi)(0) + O(N^{-\frac{3}{2}+}), \tag{3.7}$$

for all $t \in [0,\delta]$.

Remark: Equation (3.7) asserts that $I_N\phi$, though not a solution of the nonlinear problem (1.1), enjoys something akin to energy conservation. If one could replace the increment $N^{-\frac{3}{2}+}$ in $E(I_N\phi)$ on the right side of (3.7) with $N^{-\alpha}$ for some $\alpha > 0$, one could repeat the argument we give below to prove global well posedness of (1.1)-(1.2) for all $s > \frac{2}{2+\alpha}$. In particular, if $E(I_N\phi)(t)$ is conserved (i.e. $\alpha = \infty$), one could show that (1.1)-(1.2) is globally well-posed when $s > 0$.

We first show that Proposition 3.1 implies (3.6). Note that the initial value problem here has a scaling symmetry, and is H^s-subcritical when $1 > s > 0, \frac{1}{2}$ and $n = 2, 3$, respectively. That is, if ϕ is a solution to (1.1), so too

$$\phi^{(\lambda)}(x,t) \equiv \frac{1}{\lambda}\phi(\frac{x}{\lambda}, \frac{t}{\lambda^2}). \tag{3.8}$$

Using (3.3), the following energy can be made arbitrarily small by taking λ large,

$$E(I_N\phi_0^{(\lambda)}) \leq \left((N^{2-2s})\lambda^{-2s} + \lambda^{-2}\right) \cdot (1 + ||\phi_0||_{H^s(\mathbf{R}^2)})^4 \tag{3.9}$$

$$\leq C_0(N^{2-2s}\lambda^{-2s}) \cdot (1 + ||\phi_0||_{H^s(\mathbf{R}^2)})^4. \tag{3.10}$$

Assuming $N \gg 1$ is given[4], we choose our scaling parameter $\lambda = \lambda(N, ||\phi||_{H^s(\mathbf{R}^2)})$

$$\lambda = N^{\frac{1-s}{s}} \left(\frac{1}{2C_0}\right)^{-\frac{1}{2s}} \cdot \left(1 + ||\phi_0||_{H^s(\mathbf{R}^2)}\right)^{\frac{2}{s}} \tag{3.11}$$

so that $E(I_N\phi_0^{(\lambda)}) \leq \frac{1}{2}$. We may now apply Proposition 3.1 to the scaled initial data $\phi_0^{(\lambda)}$, and in fact may reapply this Proposition until the size of $E(I_N\phi^{(\lambda)})(t)$ reaches 1, that is at least $C_1 \cdot N^{\frac{3}{2}-}$ times. Hence

$$E(I_N\phi^{(\lambda)})(C_1 N^{\frac{3}{2}-}\delta) \sim 1. \tag{3.12}$$

Given any $T_0 \gg 1$, we establish the polynomial growth (3.6) from (3.12) by first choosing our parameter $N \gg 1$ so that

$$T_0 \sim \frac{N^{\frac{3}{2}-}}{\lambda^2} C_1 \cdot \delta \sim N^{\frac{7s-4}{2s}-}, \tag{3.13}$$

[4]The parameter N will be chosen shortly.

where we've kept in mind (3.11). Note the exponent of N on the right of (3.13) is positive provided $s > \frac{4}{7}$, hence the definition of N makes sense for arbitrary T_0. In two space dimensions,

$$E(I_N\phi)(t) = \lambda^2 E(I_N\phi^{(\lambda)})(\lambda^2 t).$$

We use (3.11), (3.12), and (3.13) to conclude that for $T_0 \gg 1$,

$$E(I_N\phi)(T_0) \leq C_2 T_0^{\frac{1-s}{\frac{7}{4}s-1}+}, \tag{3.14}$$

where N is chosen as in (3.13) and $C_2 = C_2(||\phi_0||_{H^s(\mathbf{R}^2)}, \delta)$. Together with (3.4), the bound (3.14) establishes the desired polynomial bound (3.5).

It remains then to prove Proposition 3.1. We will need the following modified version of the usual local existence theorem, wherein we control for small times the smoothed solution in the $X^\delta_{1,\frac{1}{2}+}$ norm.

Proposition 3.2. *Assume $\frac{4}{7} < s < 1$ and we are given data for the problem* (1.1)-(1.2) *with $E(I\phi_0) \leq 1$. Then there is a constant $\delta = \delta(||\phi_0||_{L^2(\mathbf{R}^2)})$ so that the solution ϕ obeys the following bound on the time interval* $[0, \delta]$,

$$||I\phi||_{X^\delta_{1,\frac{1}{2}+}} \lesssim 1. \tag{3.15}$$

Proof. We mimic the typical iteration argument showing local existence. We will need the following three estimates involving the $X_{s,\delta}$ spaces (2.1) and functions $F(x,t), f(x)$. (Throughout this section, the implicit constants in the notation $\lesssim$ are independent of δ.)

$$||S(t)f||_{X^\delta_{1,\frac{1}{2}+}} \lesssim ||f||_{H^1(\mathbf{R}^2)}, \tag{3.16}$$

$$\left\| \int_0^t S(t-\tau)F(x,\tau)d\tau \right\|_{X_{1,\frac{1}{2}+}} \lesssim ||F||_{X^\delta_{1,-\frac{1}{2}+}}, \tag{3.17}$$

$$||F||_{X^\delta_{1,-b}} \lesssim \delta^P ||F||_{X^\delta_{1,-\beta}}, \tag{3.18}$$

where in (3.18) we have $0 < \beta < b < \frac{1}{2}$, and $P = \frac{1}{2}(1 - \frac{\beta}{b}) > 0$. The bounds (3.16), (3.17) are analogous to estimates (3.13), (3.15) in [20]. As for (3.18), by duality it suffices to show

$$||F||_{X^\delta_{-1,\beta}} \lesssim \delta^P ||F||_{X^\delta_{-1,b}}.$$

Interpolation[5] gives

$$||F||_{X^\delta_{-1,\beta}} \lesssim ||F||^{(1-\frac{\beta}{b})-}_{X^\delta_{-1,0}} \cdot ||F||^{\frac{\beta}{b}}_{X^\delta_{-1,b}}.$$

As $b \in (0, \frac{1}{2})$, arguing exactly as on page 771 of [7],

$$||F||_{X^\delta_{-1,0}} \lesssim \delta^{\frac{1}{2}} ||F||_{X^\delta_{-1,b}},$$

and (3.18) follows.

[5]The argument here actually involves Lemma 3.2 of [20]. We thank S. Selberg for pointing this out to us.

Duhamel's principle and (3.16)- (3.18) give us

$$
\begin{aligned}
||I\phi||_{X^{\delta}_{1,\frac{1}{2}+}} &= \left\| S(t)(I\phi_0) + \int_0^t S(t-\tau) I(\phi\bar{\phi}\phi)(\tau) d\tau \right\|_{X^{\delta}_{1,\frac{1}{2}+}} \\
&\lesssim ||I\phi_0||_{H^1(\mathbf{R}^2)} + ||I(\phi\overline{\phi}\phi)||_{X^{\delta}_{1,-\frac{1}{2}+}} \\
&\lesssim ||I\phi_0||_{H^1(\mathbf{R}^2)} + \delta^{\epsilon} ||I(\phi\overline{\phi}\phi)||_{X^{\delta}_{1,-\frac{1}{2}++}}, \qquad (3.19)
\end{aligned}
$$

where $-\frac{1}{2}++$ is a real number slightly larger than $-\frac{1}{2}+$ and $\epsilon > 0$. By the definition of the restricted norm (2.2),

$$
||I\phi||_{X^{\delta}_{1,\frac{1}{2}+}} \lesssim ||I\phi_0||_{H^1(\mathbf{R}^2)} + \delta^{\epsilon} ||I(\psi\overline{\psi}\psi)||_{X_{1,-\frac{1}{2}++}}, \qquad (3.20)
$$

where the function ψ agrees with ϕ for $t \in [0, \delta]$, and

$$
||I\phi||_{X^{\delta}_{1,\frac{1}{2}+}} \sim \quad ||I\psi||_{X_{1,\frac{1}{2}+}}. \qquad (3.21)
$$

We will show shortly that

$$
||I(\psi\overline{\psi}\psi)||_{X_{1,-\frac{1}{2}++}} \lesssim \quad ||I\psi||^3_{X_{1,\frac{1}{2}+}}. \qquad (3.22)
$$

Setting then $Q(\delta) \equiv ||I\phi(t)||_{X^{\delta}_{1,\frac{1}{2}+}}$, the bounds (3.19), (3.21) and (3.22) yield

$$
Q(\delta) \lesssim ||I\phi_0||_{H^1(\mathbf{R}^2)} + \delta^{\epsilon} (Q(\delta))^3. \qquad (3.23)
$$

Note

$$
||I\phi_0||_{H^1(\mathbf{R}^2)} \lesssim (E(I\phi_0))^{\frac{1}{2}} + ||\phi_0||_{L^2(\mathbf{R}^2)} \lesssim 1 + ||\phi_0||_{L^2(\mathbf{R}^2)}. \qquad (3.24)
$$

As Q is continuous in the variable δ, a bootstrap argument yields (3.15) from (3.23), (3.24).

It remains to show (3.22). Using the interpolation lemma of [11], it suffices to show

$$
||\psi\bar{\psi}\psi||_{X_{s,-\frac{1}{2}++}} \lesssim ||\psi||^3_{X_{s,\frac{1}{2}+}}, \qquad (3.25)
$$

for all $\frac{4}{7} < s < 1$. By duality and a "Leibniz" rule[6], (3.25) follows from

$$
\left| \int_{\mathbf{R}} \int_{\mathbf{R}^2} (\langle \nabla \rangle^s \phi_1) \overline{\phi_2} \phi_3 \phi_4 dx dt \right| \lesssim ||\phi_1||_{X_{s,\frac{1}{2}+}} \cdot ||\phi_2||_{X_{s,\frac{1}{2}+}} \cdot ||\phi_3||_{X_{s,\frac{1}{2}+}} ||\phi_4||_{X_{0,\frac{1}{2}--}}. \qquad (3.26)
$$

Note that since the factors in the integrand on the left here will be taken in absolute value, the relative placement of complex conjugates is irrelevant. Use Hölder's inequality on the left side of (3.26), taking the factors in, respectively, $L^4_{x,t}, L^4_{x,t}, L^6_{x,t}$ and $L^3_{x,t}$. Using a Strichartz inequality,

$$
\begin{aligned}
||\langle \nabla \rangle^s \phi_1||_{L^4_{x,t}(\mathbf{R}^{2+1})} &\lesssim ||\langle \nabla \rangle^s \phi_1||_{X_{0,\frac{1}{2}+}} \\
&= ||\phi_1||_{X_{s,\frac{1}{2}+}},
\end{aligned}
$$

and

$$
\begin{aligned}
||\phi_2||_{L^4_{x,t}(\mathbf{R}^{2+1})} &\lesssim ||\phi_2||_{X_{0,\frac{1}{2}+}} \\
&\lesssim ||\phi_2||_{X_{s,\frac{1}{2}+}}.
\end{aligned}
$$

[6]By this, we mean the operator $\langle D \rangle^s$ can be distributed over the product by taking Fourier transform and using $\langle \xi_1 + \ldots \xi_4 \rangle^s \lesssim \langle \xi_1 \rangle^s + \ldots \langle \xi_4 \rangle^s$.

The bound for the third factor uses Sobolev embedding and the $L_t^6 L_x^3$ Strichartz estimate,

$$\begin{aligned} \|\phi_3\|_{L_t^6 L_x^6(\mathbf{R}^{2+1})} &\lesssim \|\langle\nabla\rangle^{\frac{1}{3}}\phi_3\|_{L_t^6 L_x^3(\mathbf{R}^{2+1})} \\ &\lesssim \|\langle\nabla\rangle^{\frac{1}{3}}\phi_3\|_{X_{0,\frac{1}{2}+}} \\ &\leq \|\phi_3\|_{X_{s,\frac{1}{2}+}}. \end{aligned}$$

It remains to bound $\|\phi_4\|_{L^3(\mathbf{R}^{2+1})}$. Interpolating between $\|\phi_4\|_{L_t^2 L_x^2} \leq \|\phi_4\|_{X_{0,0}}$ and the Strichartz estimate $\|\phi_4\|_{L_t^4 L_x^4} \lesssim \|\phi_4\|_{X_{0,\frac{1}{2}+}}$ yields

$$\|\phi_4\|_{L_t^3 L_x^3} \lesssim \|\phi_4\|_{X_{0,\frac{1}{2}--}}.$$

This completes the proof of (3.26), and hence Proposition 3.2. $\square$

Proof of Proposition 3.1. The usual energy (1.4) is shown to be conserved by differentiating in time, integrating by parts, and using the equation (1.1),

$$\begin{aligned} \partial_t E(\phi) &= \mathrm{Re}\int_{\mathbf{R}^2} \overline{\phi_t}(|\phi|^2\phi - \Delta\phi)dx \\ &= \mathrm{Re}\int_{\mathbf{R}^2} \overline{\phi_t}(|\phi|^2\phi - \Delta\phi - i\phi_t)dx \\ &= 0. \end{aligned}$$

We follow the same strategy to estimate the growth of $E(I\phi)(t)$,

$$\begin{aligned} \partial_t E(I\phi)(t) &= \mathrm{Re}\int_{\mathbf{R}^2} \overline{I(\phi)_t}(|I\phi|^2 I\phi - \Delta I\phi - iI\phi_t)dx \\ &= \mathrm{Re}\int_{\mathbf{R}^2} \overline{I(\phi)_t}(|I\phi|^2 I\phi - I(|\phi|^2\phi))dx, \end{aligned}$$

where in the last step we've applied I to (1.1). When we integrate in time and apply the Parseval formula[7] it remains for us to bound

$$E(I\phi(\delta)) - E(I\phi(0)) = \int_0^\delta \int_{\sum_{j=1}^4 \xi_j = 0} \left(1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)\cdot m(\xi_3)\cdot m(\xi_4)}\right) \widehat{\overline{I\partial_t\phi}}(\xi_1)\widehat{I\phi}(\xi_2)\widehat{\overline{I\phi}}(\xi_3)\widehat{I\phi}(\xi_4). \tag{3.27}$$

The reader may ignore the appearance of complex conjugates here and in the sequel, as they have no impact on the availability of estimates. (See e.g. Lemma 2.1 above.) We include the complex conjugates for completeness.

We use the equation (1.1) to substitute for $\partial_t I(\phi)$ in (3.27). Our aim is to show that

$$\mathrm{Term}_1 + \mathrm{Term}_2 \lesssim N^{-\frac{3}{2}+}, \tag{3.28}$$

[7]That is, $\int_{\mathbf{R}^n} f_1(x)f_2(x)f_3(x)f_4(x)dx = \int_{\xi_1+\xi_2+\xi_3+\xi_4=0} \hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\hat{f}_3(\xi_3)\hat{f}_4(\xi_4)$ where $\int_{\sum_i \xi_i=0}$ here denotes integration with respect to the hyperplane's measure
$\delta_0(\xi_1+\xi_2+\xi_3+\xi_4)d\xi_1 d\xi_2 d\xi_3 d\xi_4$, with δ_0 the one dimensional Dirac mass.

where the two terms on the left are

$$\text{Term}_1 \equiv \left| \int_0^\delta \int_{\sum_{i=1}^4 \xi_i = 0} \left(1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)}\right) \widehat{(\Delta \overline{I\phi})}(\xi_1) \cdot \widehat{I\phi}(\xi_2) \cdot \widehat{\overline{I\phi}}(\xi_3) \cdot \widehat{I\phi}(\xi_4) \right| \tag{3.29}$$

$$\text{Term}_2 \equiv \left| \int_0^\delta \int_{\sum_{i=1}^4 \xi_i = 0} \left(1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)}\right) \widehat{(\overline{I(|\phi|^2\phi)})}(\xi_1) \cdot \widehat{I\phi}(\xi_2) \cdot \widehat{\overline{I\phi}}(\xi_3) \cdot \widehat{I\phi}(\xi_4) \right|. \tag{3.30}$$

In both cases we break ϕ into a sum of dyadic constituents ψ_j, each with frequency support $\langle \xi \rangle \sim 2^j$, $j = 0, \ldots$..

For both Term_1 and Term_2 we'll pull the symbol

$$1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} \tag{3.31}$$

out of the integral, estimating it pointwise in absolute value, using two different strategies depending on the relative sizes of the frequencies involved. After so bounding the factor (3.31), the remaining integrals in (3.29), (3.30), involving the pieces ψ_i of ϕ, are estimated by reversing the Plancherel formula[8] and using duality, Hölder's inequality, and Strichartz estimates. We can sum over the all frequency pieces ψ_i since our bounds decay geometrically in these frequencies. We suggest that the reader at first ignore this summation issue, and so ignore on first reading the appearance below of all factors such as N_i^{0-} which we include only to show explicitly why our frequency interaction estimates sum. The main goal of the analysis is to establish the decay of $N^{-\frac{3}{2}+}$ in each class of frequency interactions below.

Consider first Term_1. By Proposition 3.2,

$$\begin{aligned} \|\Delta(I\phi)\|_{X^\delta_{-1,\frac{1}{2}+}} &\le \|I\phi\|_{X^\delta_{1,\frac{1}{2}+}} \\ &\lesssim 1. \end{aligned}$$

Hence we conclude $\text{Term}_1 \lesssim N^{-\frac{3}{2}+}$ once we show

$$\begin{aligned} &\left| \int_0^\delta \int_{\sum_{i=1}^4 \xi_i = 0} \left(1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)\cdot m(\xi_3)\cdot m(\xi_4)}\right) \widehat{\overline{\phi_1}}(\xi_1)\widehat{\phi_2}(\xi_2)\widehat{\overline{\phi_3}}(\xi_3)\widehat{\phi_4}(\xi_4) \right| \\ &\qquad \lesssim N^{-\frac{3}{2}+}(N_1N_2N_3N_4)^{0-}\|\phi_1\|_{X_{-1,\frac{1}{2}+}} \cdot \|\phi_2\|_{X_{1,\frac{1}{2}+}} \cdot \|\phi_3\|_{X_{1,\frac{1}{2}+}} \cdot \|\phi_4\|_{X_{1,\frac{1}{2}+}}, \end{aligned} \tag{3.32}$$

for any functions ϕ_i, $i = 1, \ldots, 4$ with positive spatial Fourier transforms supported on

$$\langle \xi \rangle \sim 2^{k_i} \quad \equiv N_i, \tag{3.33}$$

for some $k_i \in \{0, 1, \ldots\}$. (Note that we are not decomposing the frequencies $|\xi| \le 1$ here. In the three dimensional argument we'll need to do this.) The inequality (3.32) implies our desired bound (3.28) for Term_1 once we sum over all dyadic pieces ψ_j.

By the symmetry of the multiplier (3.31) in ξ_2, ξ_3, ξ_4, and the fact that the refined Strichartz estimate (2.6) allows complex conjugates on either factor, we may assume for the remainder of this proof that

$$N_2 \ge N_3 \ge N_4. \tag{3.34}$$

Note too that $\sum_{i=1}^4 \xi_i = 0$ in the integration of (3.32) so that $N_1 \lesssim N_2$. Hence it is sufficient to obtain a decay factor of $N^{-\frac{3}{2}+}N_2^{0-}$ on the right hand side of (3.32). We now split the different frequency interactions into three cases, according to the size of the parameter N in comparison to the N_i.

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[8]Assuming, as we may, that the spatial Fourier transform of ϕ is always positive.

Term$_1$, Case 1: $N \gg N_2$. According to (3.2), the symbol (3.31) is in this case identically zero and the bound (3.32) holds trivially.

Term$_1$, Case 2: $N_2 \gtrsim N \gg N_3 \geq N_4$. Since $\sum_i \xi_i = 0$, we have here also $N_1 \sim N_2$. By the mean value theorem,

$$\left| \frac{m(\xi_2) - m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)} \right| \lesssim \frac{|\nabla m(\xi_2) \cdot (\xi_3 + \xi_4)|}{m(\xi_2)} \lesssim \frac{N_3}{N_2}. \tag{3.35}$$

This pointwise bound together with Plancherel's theorem and (2.6) yield

$$\text{Left Side of (3.32)} \leq \frac{N_3}{N_2} ||\phi_1 \phi_3||_{L^2([0,\delta]\times \mathbf{R}^2])} ||\phi_2 \phi_4||_{L^2([0,\delta]\times \mathbf{R}^2)} \tag{3.36}$$

$$\leq \frac{N_3 N_3^{\frac{1}{2}} N_4^{\frac{1}{2}}}{N_2 N_1^{\frac{1}{2}} N_2^{\frac{1}{2}}} \prod_i ||\phi_i||_{X^\delta_{0,\frac{1}{2}+}}. \tag{3.37}$$

Comparing (3.32) with (3.37) it remains only to show that

$$\frac{N_3 N_3^{\frac{1}{2}} N_4^{\frac{1}{2}} \langle N_1 \rangle}{N_2 N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} N_2 \langle N_3 \rangle \langle N_4 \rangle} \lesssim N^{-\frac{3}{2}+} N_2^{0-},$$

which follows immediately from our assumptions $N_1 \sim N_2 \gtrsim N \gg N_3 \geq N_4$.

Term$_1$, Case 3: $N_2 \geq N_3 \gtrsim N$. We use in this instance a trivial pointwise bound on the symbol,

$$\left| 1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2) m(\xi_3) m(\xi_4)} \right| \lesssim \frac{m(\xi_1)}{m(\xi_2) m(\xi_3) m(\xi_4)}. \tag{3.38}$$

When estimating the remainder of the integrand on the left of (3.32), break the interactions into two subcases, depending on which frequency is comparable to N_2.

Case 3(a): $N_1 \sim N_2 \geq N_3 \gtrsim N$. We aim for

$$\frac{m(N_1)}{m(N_2) m(N_3) m(N_4)} \cdot \left| \int_0^\delta \int_{\sum_{i=1}^4 \xi_i = 0} \widehat{\overline{\phi_1}} \widehat{\phi_2} \widehat{\overline{\phi_3}} \widehat{\phi_4} \right| \lesssim \frac{N^{-\frac{3}{2}+} N_2^{0-} N_2 N_3 \langle N_4 \rangle}{N_1} \prod_{i=1}^4 ||\phi_i||_{X^\delta_{0,\frac{1}{2}+}}.$$

Pairing $\overline{\phi_1} \cdot \phi_4$ and $\phi_2 \cdot \overline{\phi_3}$ in L^2 and applying (2.6), it remains to show

$$\frac{m(N_1) N_4^{\frac{1}{2}} N_3^{\frac{1}{2}}}{m(N_2) m(N_3) m(N_4) N_1^{\frac{1}{2}} N_2^{\frac{1}{2}}} \lesssim N^{-\frac{3}{2}+} \frac{N_2^{1-} N_3 \langle N_4 \rangle}{N_1},$$

or

$$\frac{N^{\frac{3}{2}+} N_2^{0-}}{m(N_3) m(N_4) N_2 N_3^{\frac{1}{2}} \langle N_4 \rangle^{\frac{1}{2}}} \lesssim 1. \tag{3.39}$$

When estimating such fractions here and in the sequel, we frequently use two trivial observations[9]: for any $p > \frac{3}{7}$, the function $m(x) x^p$ is increasing; and $m(x) \langle x \rangle^p$ is bounded below. For example, in the denominator of (3.39), $m(N_4) \langle N_4 \rangle^{\frac{1}{2}} \gtrsim 1$ and $m(N_3) N_3^{\frac{1}{2}} \gtrsim m(N) N^{\frac{1}{2}} = N^{\frac{1}{2}}$. After these observations one quickly concludes that (3.39) holds.

---

[9]Alternatively, use (3.2) to write out the value of m explicitly.

Case 3(b): $N_2 \sim N_3 \gtrsim N$. In this case we also know $N_1 \lesssim N_2$, since the frequencies ξ_i must sum to zero. Argue as above, now pairing $\phi_1\phi_2$ and $\phi_3\phi_4$ in L^2. The desired bound (3.32) will follow from

$$\frac{m(N_1)N_1^{\frac{1}{2}}N_4^{\frac{1}{2}}}{m(N_2)m(N_3)m(N_4)N_2^{\frac{1}{2}}N_3^{\frac{1}{2}}} \lesssim N^{-\frac{3}{2}+}\frac{N_2^{1-}N_3\langle N_4\rangle}{\langle N_1\rangle},$$

or, after cancelling powers of N_1 in the numerator with powers of N_2 in the denominator,

$$\frac{m(N_1)N^{\frac{3}{2}-}N_2^{0+}}{m(N_2)m(N_3)m(N_4)N_3^{\frac{1}{2}}N_2\langle N_4\rangle^{\frac{1}{2}}} \lesssim 1. \tag{3.40}$$

Using $m(N_4)\langle N_4\rangle^{\frac{1}{2}} \gtrsim 1$ and that both $m(N_2)N_2^{\frac{1}{2}}$, $m(N_3)N_3^{\frac{1}{2}} \gtrsim m(N)N^{\frac{1}{2}} = N^{\frac{1}{2}}$, we get (3.40). This completes the proof of (3.32), and the bound for the contribution of Term_1 in (3.28).

We turn to the bound (3.28) for Term_2 (3.30). As in our previous discussion of Term_1, it suffices to show

$$\left|\int_0^\delta \int_{\sum_{i=1}^6 \xi_i=0}\left(1-\frac{m(\xi_4+\xi_5+\xi_6)}{m(\xi_4)m(\xi_5)m(\xi_6)}\right)P_{N_{123}}\widehat{\overline{I(\phi_1\phi_2\phi_3)}}(\xi_1+\xi_2+\xi_3)\widehat{I\phi_4}(\xi_4)\widehat{\overline{I\phi_5}}(\xi_5)\widehat{I\phi_6}(\xi_6)\right| \tag{3.41}$$
$$\lesssim N^{-\frac{3}{2}+}N_4^{0-}\prod_{i=1}^{6}||I\phi_i||_{X_{1,\frac{1}{2}+}},$$

where as above, $0 \leq \widehat{\phi_i}(\xi_i)$ is supported for $|\xi_i| \sim N_i = 2^{k_i}$, and without loss of generality,

$$N_4 \geq N_5 \geq N_6, \text{ and } N_4 \gtrsim N, \tag{3.42}$$

the latter assumption since otherwise the symbol on the left of (3.41) vanishes. In (3.41) we have written $P_{N_{123}}$ for the projection onto functions supported in the N_{123} dyadic spatial frequency shell. The decay factor on the right of (3.41) allows us to sum in N_4, N_5, N_6, and N_{123}, which suffices as we do not dyadically decompose that part of Term_2 represented here by $\phi_i, i = 1,2,3$. We pointwise bound the symbol on the left of (3.41) in the obvious way

$$\left|1-\frac{m(\xi_4+\xi_5+\xi_6)}{m(\xi_4)m(\xi_5)m(\xi_6)}\right| \lesssim \frac{m(N_{123})}{m(N_4)m(N_5)m(N_6)}$$

and as before, we undo the Plancherel formula. After applying Hölder's inequality, it suffices to show

$$\frac{m(N_{123})}{m(N_4)m(N_5)m(N_6)}\cdot||P_{N_{123}}I(\phi_1\phi_2\phi_3)||_{L_t^2L_x^2}\cdot||I\phi_4||_{L_t^4L_x^4}\cdot||I\phi_5||_{L_t^4L_x^4}\cdot||I\phi_6||_{L_t^\infty L_x^\infty} \tag{3.43}$$
$$\lesssim N^{-\frac{3}{2}+}N_4^{0-}\prod_{i=1}^{6}||I\phi_i||_{X_{1,\frac{1}{2}+}}.$$

To this end we'll use

Lemma 3.3. *Suppose the functions ϕ_i, $i = 1,\dots 6$ as above. Then,*

$$||P_{N_{123}}I(\phi_1\phi_2\phi_3)||_{L_t^2L_x^2} \lesssim \frac{1}{\langle N_{123}\rangle}\prod_{i=1}^{3}||I\phi_i||_{X_{1,\frac{1}{2}+}}, \tag{3.44}$$

$$||I\phi_j||_{L_t^4L_x^4} \lesssim \frac{1}{\langle N_j\rangle}||I\phi_j||_{X_{1,\frac{1}{2}+}} \quad j = 4,5, \tag{3.45}$$

$$||I\phi_6||_{L_t^\infty L_x^\infty} \lesssim ||I\phi_6||_{X_{1,\frac{1}{2}+}}. \tag{3.46}$$

Proof. For (3.44), it suffices to prove

$$||\langle\nabla\rangle P_{N_{123}} I(\phi_1\phi_2\phi_3)||_{L_t^2L_x^2} \lesssim \prod_{i=1}^{3} ||I\phi_i||_{X_{1,\frac{1}{2}+}}. \tag{3.47}$$

(See Section 2 above for notation). The operator $\langle\nabla\rangle I$ obeys a Leibniz rule. Using Hölder's inequality on a typical resulting term,

$$||P_{N_{123}}\left((\langle\nabla\rangle I(\phi_1))\phi_2\phi_3\right)||_{L_t^2L_x^2} \lesssim ||\langle\nabla\rangle I(\phi_1)||_{L^4_{x,t}}||\phi_2||_{L^8_{x,t}}||\phi_3||_{L^8_{x,t}}. \tag{3.48}$$

By Sobolev's inequality and a $L_t^8L_x^{\frac{8}{3}}$ Strichartz estimate (2.5),

$$\begin{aligned}
||\phi_2||_{L^8_{t,x}} &\lesssim ||\langle\nabla\rangle^{\frac{1}{2}}\phi_2||_{L_t^8L_x^{\frac{8}{3}}} \\
&\lesssim ||\langle\nabla\rangle^{\frac{1}{2}}\phi_2||_{X_{0,\frac{1}{2}+}} \\
&\lesssim ||\phi_2||_{X_{1,\frac{1}{2}+}}
\end{aligned} \tag{3.49}$$

and similarly for the ϕ_3 factor on the right of (3.48). Applying the $L^4_{x,t}$ Strichartz estimate,

$$||\langle\nabla\rangle I\phi_1||_{L^4_{x,t}} \lesssim ||I\phi_1||_{X_{1,\frac{1}{2}+}}. \tag{3.50}$$

Together, (3.48) - (3.50) yield (3.44).

The bounds (3.45) follow immediately from the $L^4_{t,x}$ Strichartz estimate as in (3.50). The estimate (3.46) is seen using Sobolev embedding, the fact that ϕ_6 is frequency localized, and the $L_t^\infty L_x^2$ Strichartz bound,

$$\begin{aligned}
||I\phi_6||_{L^\infty_{x,t}} &\lesssim ||\langle\nabla\rangle I\phi_6||_{L_t^\infty L_x^2} \\
&\lesssim ||I\phi_6||_{X_{1,\frac{1}{2}+}}.
\end{aligned}$$

□

Together, (3.43) and Lemma 3.3 leave us to show

$$\frac{m(N_{123})\cdot N^{\frac{3}{2}-}N_4^{0+}}{m(N_4)m(N_5)m(N_6)\langle N_{123}\rangle\langle N_4\rangle\langle N_5\rangle} \lesssim 1 \tag{3.51}$$

under the assumption (3.42). We can break the frequency interactions into two cases: $N_4 \sim N_5$ and $N_4 \sim N_{123}$, since we have $\sum_{i=1}^6 \xi_i = 0$ in (3.41).

Term$_2$, Case 1; $N_4 \sim N_5; N_4 \geq N_5 \geq N_6;\ N_4 \gtrsim N$: We aim here for

$$\frac{m(N_{123})N^{\frac{3}{2}-}N_4^{0+}}{(m(N_4))^2\langle N_4\rangle^2 m(N_6)\langle N_{123}\rangle} \lesssim 1.$$

Since $m(N_4)\langle N_4\rangle^{\frac{1}{2}} \gtrsim m(N)\langle N\rangle^{\frac{1}{2}} = \langle N\rangle^{\frac{1}{2}}$ it suffices to show

$$\frac{m(N_{123})N^{\frac{1}{2}-}N_4^{0+}}{\langle N_4\rangle m(N_6)\langle N_{123}\rangle} \lesssim 1, \tag{3.52}$$

which is clear since $\langle N_{123}\rangle \geq m(N_{123})$, and

$$m(y)\langle x\rangle^{\frac{1}{2}} \gtrsim 1 \quad \text{for all} \quad 0 \leq y \leq x. \tag{3.53}$$

Term$_2$, Case 2; $N_4 \sim N_{123}; N_4 \geq N_5 \geq N_6;\ N_4 \gtrsim N$: Here we argue that

$$\frac{m(N_4) N^{\frac{3}{2}-} N_4^{0+}}{m(N_4)\langle N_4\rangle^2 m(N_5) m(N_6)\langle N_5\rangle} \lesssim \frac{N^{\frac{3}{2}-} N_4^{0+}}{m(N_5)\langle N_5\rangle^{\frac{1}{2}} m(N_6)\langle N_4\rangle^2 \langle N_5\rangle^{\frac{1}{2}}} \lesssim 1,$$

using (3.53) and our assumptions on the N_i. This completes the proof of (3.28) and hence the proof of Proposition 3.1. □

4. Proof of Theorem 1.1 in $\mathbf{R}^3$

In three space dimensions our almost conservation law takes the following form,

Proposition 4.1. *Given $s > \frac{5}{6}, N \gg 1$, and initial data $\phi_0 \in C_0^\infty(\mathbf{R}^3)$ with $E(I_N\phi_0) \leq 1$, then there exists a universal constant δ so that the solution*

$$\phi(x,t) \in C([0,\delta], H^s(\mathbf{R}^3))$$

of (1.1)-(1.2) *satisfies*

$$E(I_N\phi)(t) = E(I_N\phi)(0) + O(N^{-1+}), \tag{4.1}$$

for all $t \in [0,\delta]$.

The norm $||\phi(t,\cdot)||_{L^2(\mathbf{R}^3)}$ is supercritical with respect to the scaling (3.8). Hence, aside from the L^2 conservation (1.3) we will avoid using this quantity in the proof of the three dimensional result. Beside the technical issues introduced by scaling the L^2 norm, our proof of Theorem 1.1 for $n = 3$ follows very closely the $n = 2$ arguments of Section 3.

We begin with the fact that Proposition 4.1 implies Theorem 1.1 with $n = 3$. Recall that it suffices to show the $H^s(\mathbf{R}^3)$ norm of the solution to (1.1)- (1.2) grows polynomially in time. Recall too $\phi^{(\lambda)}$ as the scaled solution defined in (3.8). When $n = 3$, the definition of the energy (1.4) and Sobolev embedding imply

$$\begin{aligned} E(I_N\phi_0^{(\lambda)}) &= \frac{1}{2}||\nabla I_N \phi_0^{(\lambda)}||^2_{L^2(\mathbf{R}^3)} + \frac{1}{4}||I_N\phi_0^{(\lambda)}||^4_{L^4(\mathbf{R}^3)} \\ &\leq C_0 N^{2-2s}\lambda^{1-2s}(1+||\phi_0||_{H^s(\mathbf{R}^3)})^4. \end{aligned} \tag{4.2}$$

Once the parameter N is chosen, we will choose λ according to

$$\lambda = \left(\frac{1}{2C_0}\right)^{\frac{1}{1-2s}} N^{\frac{2s-2}{1-2s}}(1+||\phi_0||_{H^s(\mathbf{R}^3)})^{-\frac{4}{1-2s}}. \tag{4.3}$$

Together, (4.3) and (4.2) give $E(I_N\phi_0^{(\lambda)}) \leq \frac{1}{2}$. We can therefore apply Proposition 4.1 at least $C_1 \cdot N^{1-}$ times to give

$$E(I_N\phi^{(\lambda)})(C_1 N^{1-}\cdot\delta) \sim 1. \tag{4.4}$$

The estimate (4.4) implies $||\phi(t,\cdot)||_{H^s(\mathbf{R}^3)}$ grows at most polynomially when $\frac{5}{6} < s < 1$. This can be seen exactly as in the two dimensional case. We include the argument here for completeness.

Given any $T_0 \gg 1$, first choose $N \gg 1$ so that

$$T_0 = \frac{C_1 N^{1-}\delta}{\lambda^2} \sim N^{\left(\frac{\frac{5}{2}-3s-}{\frac{1}{2}-s}\right)}. \tag{4.5}$$

Note that the exponent of N on the right of (4.5) is positive (and hence this definition of N makes sense) precisely when $s > \frac{5}{6}$. In three space dimensions we have

$$E(I_N \phi^{(\lambda)})(\lambda^2 t) = \frac{1}{\lambda} E(I_N \phi)(t).$$

According to (4.3), (4.4), (4.5), we therefore get

$$\begin{aligned} E(I_N\phi)(T_0) &\leq \lambda E(I_N \phi^{(\lambda)})(\lambda^2 T_0) \\ &\lesssim \lambda \\ &\lesssim N^{\frac{2s-2}{1-2s}} \\ &\lesssim T_0^{\frac{1-s+}{3(s-\frac{5}{6})}}. \end{aligned}$$

According to (3.4) and (1.3), the $H^s(\mathbf{R}^3)$ norm grows with at most half this rate when $\frac{5}{6} < s < 1$,

$$||\phi||_{H^s(\mathbf{R}^3)}(T) \lesssim (1+T)^{\frac{1-s+}{6(s-\frac{5}{6})}}. \tag{4.6}$$

As in the two dimensional argument, the proof of Proposition 4.1 relies on bounds for the local-in-time H^s solution. The following analogue of Proposition 3.2 avoids the use of the norm $||\phi(\cdot,t)||_{L^2(\mathbf{R}^3)}$, which, as mentioned above, is supercritical with respect to scaling.

Proposition 4.2. *Assume $\frac{5}{6} < s < 1$ and we are given data for (1.1)-(1.2) with $E(I\phi_0) \leq 1$. Then there is a universal constant $\delta > 0$ so that the solution ϕ obeys the following bound on the time interval $[0,\delta]$,*

$$||\nabla I \phi||_{X^\delta_{0,\frac{1}{2}+}} \lesssim 1. \tag{4.7}$$

Proof. Arguing as in the proof of Proposition 3.2, it suffices to prove

$$||\nabla I(\phi\bar{\phi}\phi)||_{X^\delta_{0,-\frac{1}{2}++}} \lesssim ||\nabla I\phi||^3_{X_{0,\frac{1}{2}+}},$$

Again, the interpolation lemma in [11] allows us to assume $N=1$ in the definition (3.1) of the operator I. After applying a Leibniz rule for the operator ∇I and duality, we aim to show

$$||(\nabla I)(\phi_1)\cdot\overline{\phi_2}\cdot\phi_3\cdot\psi||_{L^1(\mathbf{R}^{3+1})} \lesssim ||\psi||_{X_{0,\frac{1}{2}--}} \prod_{i=1}^{3} ||\nabla I\phi_i||_{X_{0,\frac{1}{2}+}}. \tag{4.8}$$

Again, the complex conjugate will have no bearing on our bounds. We split the functions ϕ_j, $j=2,3$ into high and low frequency components,

$$\phi_j = \phi_j^{\text{high}} + \phi_j^{\text{low}}, \tag{4.9}$$

where

$$\begin{aligned} \operatorname{supp}\widehat{\phi_j^{\text{high}}}(\xi,t) &\subset \{|\xi| \geq \frac{1}{2}\} \\ \operatorname{supp}\widehat{\phi_j^{\text{low}}}(\xi,t) &\subset \{|\xi| \leq 1\}. \end{aligned}$$

Note that when $n=3$, homogeneous Sobolev embedding and the $L^{10}_t L^{\frac{30}{13}}_x$ Strichartz estimate give

$$\begin{aligned} ||\phi||_{L^{10}_t L^{10}_x(\mathbf{R}^{3+1})} &\lesssim ||\nabla\phi||_{L^{10}_t L^{\frac{30}{13}}_x(\mathbf{R}^{3+1})} \\ &\lesssim ||\nabla\phi||_{X_{0,\frac{1}{2}+}}. \end{aligned} \tag{4.10}$$

Consider first the low frequency components on the left of (4.8). Apply Holder's inequality with the factors in $L^{\frac{10}{3}}_{x,t}, L^{10}_{x,t}, L^{10}_{x,t}$, and $L^2_{x,t}$ respectively. The $L^{\frac{10}{3}}_{x,t}$ Strichartz estimate along with (4.10) give,

$$||(\nabla I)(\phi_1) \cdot \overline{\phi_2^{\text{low}}} \cdot \phi_3^{\text{low}} \cdot \psi||_{L^1(\mathbf{R}^{3+1})} \lesssim ||\psi||_{X_{0,0}} ||\nabla I \phi_1||_{X_{0,\frac{1}{2}+}} \prod_{i=2}^{3} ||\nabla \phi_j^{\text{low}}||_{X_{0,\frac{1}{2}+}}.$$

Together with the fact that $\phi_j^{\text{low}} = I\phi_j^{\text{low}}$, this bound accounts for part of the low frequency contributions of ϕ_2, ϕ_3 in (4.8). A typical contribution which remains to be bounded is

$$||\nabla(I\phi_1)\overline{\phi_2^{\text{high}}}\phi_3^{\text{high}}\psi||_{L^1_{x,t}(\mathbf{R}^{3+1})}.$$

Recall the Strichartz estimate,

$$||\psi||_{L^{\frac{10}{3}}_{x,t}(\mathbf{R}^{3+1})} \lesssim ||\psi||_{X_{0,\frac{1}{2}+}}. \tag{4.11}$$

Interpolating between (4.11) and the trivial bound $||\psi||_{L^2_{x,t}(\mathbf{R}^{3+1})} \lesssim ||\psi||_{X_{0,0}}$ gives

$$||\psi||_{L^3_{x,t}(\mathbf{R}^{3+1})} \lesssim ||\psi||_{X_{0,\frac{1}{2}--}}. \tag{4.12}$$

Using (4.12) and Holder's inequality on the left of (4.8), we aim to show

$$||\nabla(I\phi_1)\phi_2^{\text{high}}\phi_3^{\text{high}}||_{L^{\frac{3}{2}}_{x,t}(\mathbf{R}^{3+1})} \lesssim \prod_{i=1}^{3} ||\nabla I\phi_i||_{X_{0,\frac{1}{2}+}}. \tag{4.13}$$

Since we've reduced to the case $N = 1$, we note $I^{-1} = \langle\nabla\rangle^g$, where

$$g \equiv 1 - s \quad \in (0, \frac{1}{6}) \tag{4.14}$$

is the gap between s and 1. We may therefore rewrite our desired estimate as

$$||\nabla(I\phi_1)(\langle\nabla\rangle^g I\overline{\phi_2^{\text{high}}})(\langle\nabla\rangle^g I\phi_3^{\text{high}})||_{L^{\frac{3}{2}}_{x,t}(\mathbf{R}^{3+1})} \leq \prod_{i=1}^{3} ||\nabla I\phi_i||_{X_{0,\frac{1}{2}+}}. \tag{4.15}$$

But this estimate follows after taking the factors on the left in $L^{\frac{10}{3}}_{x,t}$, $L^{\frac{60}{11}}_{x,t}$, $L^{\frac{60}{11}}_{x,t}$, respectively and using Hölders inequality. The first resulting factor is bounded using the $L^{\frac{10}{3}}_{x,t}$ Strichartz estimate. As for the second two factors, Sobolev embedding, the bounds (4.14) on g, and the $L^{\frac{60}{11}}_{t,x}{}^{\frac{45}{17}}$ Strichartz estimate yield for $j = 2, 3$,

$$\begin{aligned} ||\langle\nabla\rangle^g I\phi_j^{\text{high}}||_{L^{\frac{60}{11}}_{x,t}(\mathbf{R}^{3+1})} &\lesssim ||\langle\nabla\rangle^{1-g}\langle\nabla\rangle^g I\phi_j^{\text{high}}||_{L^{\frac{60}{11}}_{t,x}{}^{\frac{45}{17}}} \\ &\lesssim ||\nabla I\phi_j^{\text{high}}||_{X_{0,\frac{1}{2}+}}. \end{aligned}$$

The case where $\phi_2^{\text{low}}\phi_3^{\text{high}}$ appears on the left of (4.8) is handled similarly, using a homogeneous Sobolev embedding to bound the ϕ_2^{low} term. □

Proof of Proposition 4.1. Arguing as in the two dimensional result leaves us to show

$$\text{Term}_1 + \text{Term}_2 \lesssim N^{-1++}, \tag{4.16}$$

where the two terms on the left are as before, (3.29), (3.30). We will have to pay closer attention here than in $\mathbf{R}^2$ when we sum the various dyadic components of this estimate. The fact that we only control inhomogeneous norms (4.7) forces us to decompose the frequencies $|\xi| \leq 1$ as well.

Considering first Term_1, it follows from the definition of the $X_{s,b}$ norms (2.1) that

$$||\Delta I\phi||_{X^\delta_{-1,\frac{1}{2}+}} \lesssim ||\nabla I\phi||_{X^\delta_{0,\frac{1}{2}+}}. \tag{4.17}$$

We conclude $\text{Term}_1 \lesssim N^{-1++}$ once we prove

$$\left| \int_0^\delta \int_{\sum_{i=1}^4 \xi_i = 0} \left(1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)\cdot m(\xi_3)\cdot m(\xi_4)}\right) \widehat{\phi_1}(\xi_1)\widehat{\phi_2}(\xi_2)\widehat{\phi_3}(\xi_3)\widehat{\phi_4}(\xi_4)\right| \tag{4.18}$$
$$\lesssim \quad N^{-1++}C(N_1,N_2,N_3,N_4)||\phi_1||_{X_{-1,\frac{1}{2}+}} \cdot \prod_{j=2}^4 ||\nabla\phi_j||_{X_{0,\frac{1}{2}+}},$$

for sufficiently small $C(N_1,N_2,N_3,N_4)$ and for any smooth functions ϕ_i, $i=1,\dots 4$ with $0 \leq \widehat{\phi_i}(\xi_i)$ supported for $|\xi_i| \sim N_i \equiv 2^{k_i}$, $k_i = 0, \pm 1, \pm 2, \dots$. As before, we may assume $N_2 \geq N_3 \geq N_4$. The precise extent to which $C(N_1,N_2,N_3,N_4)$ decays in its arguments, and the fact that this decay allows us to sum over all dyadic shells, will be described below on a case-by-case basis.

In addition to the estimates (4.10), (4.11), our analysis here uses the following related bounds, all of which are quick consequences of homogeneous Sobolev embedding, Hölder's inequality in the time variable, and/or Strichartz estimates. These estimates will allow for bounds decaying in the frequencies. For a function ϕ with frequency support in the D'th dyadic shell,

$$||\phi||_{L_t^{10}L_x^{10\pm}([0,\delta]\times\mathbf{R}^3)} \lesssim D^{0\pm}||\nabla\phi||_{X_{0,\frac{1}{2}+}} \tag{4.19}$$

$$||\phi||_{L_t^{\frac{10}{3}}L_x^{\frac{10}{3}-}([0,\delta]\times\mathbf{R}^3)} \lesssim \delta^{0+}||\phi||_{X_{0,\frac{1}{2}+}} \tag{4.20}$$

$$||\phi||_{L_t^{\frac{10}{3}}L_x^{\frac{10}{3}+}([0,\delta]\times\mathbf{R}^3)} \lesssim D^{0+}||\phi||_{X_{0,\frac{1}{2}+}}. \tag{4.21}$$

Term_1, Case 1: $N \gg N_2$. Again, the symbol (3.31) is in this case identically zero and the bound (4.18) holds trivially, with $C \equiv 0$.
Term_1, Case 2: $N_2 \gtrsim N \gg N_3 \geq N_4$. We have $N_2 \sim N_1$ here as well. We will show

$$C(N_1,N_2,N_3,N_4) = N_2^{0-}N_4^{0+}. \tag{4.22}$$

With this decay factor, and the fact that we are considering here terms where $N_1 \sim N_2$, we may immediately sum over the N_1, N_2 indices. Similarly, the factor N_4^{0+} in (4.22) allows us to sum over all terms here with $N_3, N_4 \ll 1$. It remains to sum the terms where $1 \lesssim N_4 \leq N_3 \ll N$, but these introduce at worst a divergence $N^{0+}\log(N)$, which is absorbed by the decay factor N^{-1++} on the right side of (4.18).

We now show (4.18), (4.22). As before, (3.35), we bound the symbol in this case by $\frac{N_3}{N_2}$. We apply Hölder's inequality to the left side of (4.18), bounding ϕ_1, ϕ_3 in $L^{\frac{10}{3}}_{x,t}$ as in (4.11); ϕ_2 in $L_t^{\frac{10}{3}}L_x^{\frac{10}{3}-}$ as in (4.20); and ϕ_4 in $L_t^{10}L_x^{10+}$ as in (4.19) to get,

$$\text{Left Side of (4.18)} \lesssim N_4^{0+}\frac{N_3}{N_2}||\phi_1||_{X_{0,\frac{1}{2}+}}||\phi_2||_{X_{0,\frac{1}{2}+}}||\phi_3||_{X_{0,\frac{1}{2}+}}||\nabla\phi_4||_{X_{0,\frac{1}{2}+}}$$
$$\lesssim \frac{N_4^{0+}N_3N_1}{N_2\cdot N_2\cdot N_3}||\phi_1||_{X_{-1,\frac{1}{2}+}}\prod_{j=2}^4||\nabla\phi_j||_{X_{0,\frac{1}{2}+}}.$$

We conclude the bound (4.18), (4.22) for this case once we note

$$\frac{N_3N_1N^{1-}N_2^{0+}}{N_2N_2N_3} \lesssim 1,$$

which is immediate from our assumptions on the N_i.
Term$_1$, Case 3, $N_2 \geq N_3 \gtrsim N$**:** As in the two dimensional argument, we use here the straightforward bound (3.38) for the symbol. The estimate of the remainder of the integrand will break up into six different subcases, depending on which N_i is comparable to N_2, and whether or not $N_1, N_4 \ll 1$.
Case 3(a), $N_1 \sim N_2 \geq N_3 \gtrsim N; N_4 \ll 1$: We will show here

$$C(N_1, N_2, N_3, N_4) = N_4^{0+} N_3^{0-}, \tag{4.23}$$

which suffices since one may use (4.23) to sum directly in N_3, N_4, and use Cauchy-Schwarz to sum in N_1, N_2.

To establish (4.18), (4.23), estimate the ϕ_4 factor in $L_t^{10} L_x^{10+}$ using (4.19); ϕ_3 in $L_t^{\frac{10}{3}} L_x^{\frac{10}{3}-}$ as in (4.20); and ϕ_1, ϕ_2 in $L_{x,t}^{\frac{10}{3}}$ as in (4.11). It remains then to show

$$\frac{m(N_1) N_1 N^{1-} N_3^{0+}}{m(N_2) m(N_3) N_2 N_3} \lesssim 1. \tag{4.24}$$

Note that since $s \in (\frac{5}{6}, 1)$, we can use the following fact while working in three space dimensions,

$$(m(x))^{p_1} x^{p_2} \quad \text{is nondecreasing in } x \text{ when } 0 < p_1 \leq 6p_2. \tag{4.25}$$

We check (4.24) by first cancelling factors involving N_1 and N_2 from numerator and denominator,and then using (4.25),

$$\begin{aligned} \frac{m(N_1) N_1 N^{1-} N_3^{0+}}{m(N_2) m(N_3) N_2 N_3} &\lesssim \frac{N^{1-} N_3^{0+}}{m(N_3) N_3} \\ &\lesssim \frac{N^{1-} N_3^{0+}}{(m(N)) N^{1-} N_3^{0+}} \quad \lesssim 1. \end{aligned}$$

Case 3(b), $N_2 \sim N_3 \gtrsim N, N_1 \gtrsim 1, N_4 \ll 1$: Exactly as above, one shows (4.18), (4.23) holds. With this, one may sum directly in N_4, and also in N_1, N_2, N_3 using the N_3 decay in (4.23).
Case 3(c), $N_2 \sim N_3 \gtrsim N, N_1 \ll 1, N_4 \ll 1$: Here we have

$$C(N_1, N_2, N_3, N_4) = N_3^{0-} N_4^{0+} N_1^{0+}. \tag{4.26}$$

Allowing us to sum directly in all N_i. One shows (4.26) by modifying the argument in 3(a), taking ϕ_1, ϕ_2 in $L_t^{\frac{10}{3}} L_x^{\frac{10}{3}+}, L_t^{\frac{10}{3}} L_x^{\frac{10}{3}+}$, respectively.
Case 3(d), $N_1 \sim N_2 \geq N_3 \gtrsim N; N_4 \gtrsim 1$: We will show here

$$C(N_1, N_2, N_3, N_4) = N_3^{0-} N_4^{0-}, \tag{4.27}$$

allowing us to sum immediately in N_3 and N_4; summing in N_1, N_2 using Cauchy-Schwarz.

After taking the symbol out of the left side of (4.18) using (3.38), we apply Hölder's inequality as follows: estimate the ϕ_4 factor in $L_t^{10} L_x^{10-}$ using (4.19); ϕ_3 in $L_t^{\frac{10}{3}} L_x^{\frac{10}{3}+}$ as in (4.21); and ϕ_1, ϕ_2 in $L_{x,t}^{\frac{10}{3}}$ as in (4.11). We will establish (4.18), (4.27) once we show

$$\frac{m(N_1) N_1 N^{1--} N_3^{0+} N_3^{0+}}{m(N_2) m(N_3) m(N_4) N_2 N_3} \lesssim 1. \tag{4.28}$$

This is done as in the argument of Case 3(a).
Case 3(e), $N_2 \sim N_3 \gtrsim N$; $N_4 \gtrsim 1$;$N_1 \gtrsim 1$: We will show here

$$C(N_1, N_2, N_3, N_4) = N_2^{0-} N_1^{0-}, \tag{4.29}$$

allowing us to sum directly in all the N_i. The Hölder's inequality argument here takes the ϕ_1 factor in $L_t^{10} L_x^{10-}$ using (4.19); ϕ_2 in $L_t^{\frac{10}{3}} L_x^{\frac{10}{3}+}$ as in (4.21); and ϕ_3, ϕ_4 in $L_{x,t}^{\frac{10}{3}}$ as in (4.11). We will have shown (4.18), (4.29) once we show,

$$\frac{m(N_1) N_1 N^{1--} N_2^{0+} N_2^{0+}}{m(N_2) m(N_3) m(N_4) N_2 N_3} \lesssim 1. \tag{4.30}$$

This argument is by now straightforward,

$$\begin{aligned} \frac{m(N_1) N_1 N^{1--} N_2^{0+} N_2^{0+}}{m(N_2) m(N_3) m(N_4) N_2 N_3} &\lesssim \frac{N^{1--} N_2^{0+} N_2^{0+}}{m(N_2)^2 N_2} \\ &\lesssim \frac{N^{1--} N_2^{0+} N_2^{0+}}{N^{1--} N_2^{0+} N_2^{0+}} \quad \lesssim 1. \end{aligned}$$

Case 3(f), $N_2 \sim N_3 \gtrsim N$; $N_4 \gtrsim 1$;$N_1 \ll 1$: We will show here

$$C(N_1, N_2, N_3, N_4) = N_2^{0-} N_1^{0+}, \tag{4.31}$$

allowing us to sum directly in all the N_i.

The proof of (4.18), (4.31) is similar to Case (3e), now taking ϕ_1 in $L_t^{10} L_x^{10+}$ using (4.19); ϕ_2 in $L_t^{\frac{10}{3}} L_x^{\frac{10}{3}+}$ as in (4.21); and ϕ_3, ϕ_4 in $L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}$ as in (4.20).

This completes the 3-dimensional analysis of Term_1 in (4.16).

We will show $\text{Term}_2 \lesssim N^{-1+}$ using the straightforward bound (3.38) on the symbol in the case $N_2 \geq N$, and the following,

Lemma 4.3.

$$||I(\phi_1 \phi_2 \phi_3)||_{L_t^2 L_x^2([0,\delta] \times \mathbf{R}^3)} \lesssim \prod_{i=1}^{3} ||\nabla I \phi_i||_{X_{0,\frac{1}{2}+}}. \tag{4.32}$$

We postpone the proof of Lemma 4.3. As in the work for Term_1 above, the argument bounding Term_2 is complicated only by the presence of low frequencies. Our aim is to show

$$\text{Left Side of (3.41)} \lesssim C(N_{123}, N_4, N_5, N_6) \prod_{i=1}^{6} ||\nabla I \phi_i||_{X_{0,\frac{1}{2}+}}. \tag{4.33}$$

where $N_4 \geq N_5 \geq N_6$ and $N_4 \gtrsim N$, and as in the Term_1 work above, $C(N_{123}, N_4, N_5, N_6)$ decays sufficiently fast to allow us to add up the individual frequency interaction estimates to get (4.16).

We sum first the interactions involving $N_i \gtrsim 1$ for all frequencies in (4.33). In this case we'll show a decay factor of $C = N^{-1+} (N_{123} N_4 N_5 N_6)^{0-}$, allowing us to sum in each index N_i directly. Apply Hölder's inequality to the integrand on the left of (3.41), taking the factors in $L_{x,t}^2; L_{x,t}^{\frac{10}{3}}; L_{x,t}^{10}$; and $L_{x,t}^{10}$; respectively. Using (3.38), (4.32), the Strichartz estimates and (4.10) as in the Term_1 argument, it suffices to show

$$\frac{N^{1-} N_4^{0+} m(N_{123})}{N_4 m(N_4) m(N_5) m(N_6)} \lesssim 1. \tag{4.34}$$

The fact that $m(x)$ is nonincreasing in x and (4.25) give us

$$\begin{aligned}\frac{N^{1-}N_4^{0+}m(N_{123})}{N_4 m(N_4)m(N_5)m(N_6)} &\lesssim \frac{N^{1-}N_4^{0+}m(N_{123})}{N_4(m(N_4))^3}\\ &\lesssim \frac{N^{1-}N_4^{0+}m(N_{123})}{N_4^{0+}N^{1-}(m(N))^3} \quad \lesssim 1.\end{aligned}$$

The above argument is easily modified in the presence of small frequencies. We sketch these modifications here. In case $N_{123} \sim N_4$, with $N_6 \ll 1$ and possibly also $N_5 \ll 1$, we need to get factors of N_6^{0+} and possibly also N_5^{0+} on the right hand side of (4.33). We accomplish this by taking the factor $I\phi_6$ and possibly also $I\phi_5$ in $L_t^{10}L_x^{10+}$,and take the factor $I\phi_4$ in $L_t^{\frac{10}{3}}L_x^{\frac{10}{3}-}$, or possibly $L_t^{\frac{10}{3}}L_x^{\frac{10}{3}--}$.

In case $N_4 \sim N_5$, with N_{123} and/or N_6 small, a similar argument gets the necessary decay: we can take $P_{N_{123}}I(\phi_1\phi_2\phi_3)$ in $L_t^2L_x^{2+}$, and/or $I\phi_6$ in $L_t^{10}L_x^{10+}$, and take $I\phi_4$ in $L_t^{10/3}L_x^{\frac{10}{3}-}$ or $L_t^{10/3}L_x^{\frac{10}{3}--}$. □

Proof of Lemma 4.3. By the interpolation lemma in [11], we may assume $N = 1$. By Plancherel's theorem, it suffices to prove

$$(4.35) \qquad ||\phi_1 \cdot \phi_2 \cdot \phi_3||_{L^2_{x,t}([0,\delta]\times\mathbf{R}^3)} \lesssim \prod_{i=1}^{3} ||\nabla\phi_i||_{X_{0,\frac{1}{2}+}}.$$

Decomposing $\phi_i = \phi_i^{\mathrm{low}} + \phi_i^{\mathrm{high}}$ as in (4.9), we consider first the contribution when only the low frequencies interact with one another. Hölder's inequality in space-time, homogeneous Sobolev embedding, Hölder's inequality in time, and the energy estimate yield,

$$\begin{aligned}||\phi_1^{\mathrm{low}} \cdot \phi_2^{\mathrm{low}} \cdot \phi_3^{\mathrm{low}}||_{L^2_{x,t}([0,\delta]\times\mathbf{R}^3)} &= ||I\phi_1^{\mathrm{low}} \cdot I\phi_2^{\mathrm{low}} \cdot I\phi_3^{\mathrm{low}}||_{L^2_{x,t}([0,\delta]\times\mathbf{R}^3)}\\ &\lesssim \prod_{i=1}^{3} ||I\phi_i^{\mathrm{low}}||_{L^6_{x,t}([0,\delta]\times\mathbf{R}^3)}\\ &\lesssim \prod_{i=1}^{3} ||\nabla I\phi_i^{\mathrm{low}}||_{L^6_tL^2_x([0,\delta]\times\mathbf{R}^3)}\\ &\lesssim \delta^{\frac{1}{2}} \prod_{i=1}^{3} ||\nabla I\phi_i^{\mathrm{low}}||_{L^\infty_tL^2_x([0,\delta]\times\mathbf{R}^3)}\\ &\lesssim \prod_{i=1}^{3} ||\nabla I\phi_i^{\mathrm{low}}||_{X_{0,\frac{1}{2}+}}.\end{aligned}$$

A typical term whose contribution to (4.35) remains to be controlled is

$$(4.36) \qquad ||I\phi_1^{\mathrm{low}} \cdot \langle\nabla\rangle^g I\phi_2^{\mathrm{high}} \cdot \langle\nabla\rangle^g I\phi_3^{\mathrm{high}}||_{L^2_{x,t}([0,\delta]\times\mathbf{R}^3)},$$

where g is as in (4.14). Take the first factor here in $L_t^6L_x^{18}$ and each of the second two in $L_t^6L_x^{\frac{9}{2}}$ via Hölder's inequality. Note then that Sobolev embedding and the $L_t^6L_x^{\frac{18}{7}}$ Strichartz inequality give us

$$\begin{aligned}||I\phi_1^{\mathrm{low}}||_{L_t^6L_x^{18}(\mathbf{R}^{3+1})} &\lesssim ||\nabla I\phi_1^{\mathrm{low}}||_{L_t^6L_x^{\frac{18}{7}}(\mathbf{R}^{3+1})}\\ &\lesssim ||\nabla I\phi_1^{\mathrm{low}}||_{X^\delta_{0,\frac{1}{2}+}}.\end{aligned}$$

Similarly, the fact that $g \in (0, \frac{1}{6})$, Sobolev embedding, Hölder's inequality in time, and the energy estimate give us for $j = 2, 3$,

$$\begin{aligned} ||\langle\nabla\rangle^g I\phi_j^{\text{high}}||_{L_t^6 L_x^{\frac{9}{2}}([0,\delta]\times\mathbf{R}^3)} &\lesssim ||\langle\nabla\rangle I\phi_j^{\text{high}}||_{L_t^6 L_x^2([0,\delta]\times\mathbf{R}^3)} \\ &\lesssim \delta^{\frac{1}{6}} ||\nabla I\phi_j^{\text{high}}||_{L_t^\infty L_x^2(\mathbf{R}^{3+1})} \\ &\lesssim ||\nabla I\phi_j||_{X^\delta_{0,\frac{1}{2}+}}. \end{aligned}$$

This completes the proof of Lemma 4.3. □

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