arXiv:math.AP/0101261 v1 31 Jan 2001

GLOBAL WELL-POSEDNESS FOR KDV IN SOBOLEV SPACES OF NEGATIVE INDEX

J. COLLIANDER, M. KEEL, G. STAFFILANI, H. TAKAOKA, AND T. TAO

ABSTRACT. The initial value problem for the Korteweg-deVries equation on the line is shown to be globally well-posed for rough data. In particular, we show global well-posedness for initial data in $H^s(\mathbb{R})$ for $-\frac{3}{10} < s$.

1. INTRODUCTION

Consider the initial value problem for the Korteweg-deVries (KdV) equation

$$\begin{cases} \partial_t u + \partial_x^3 u + \frac{1}{2}\partial_x(u^2) = 0, & x \in \mathbb{R}, \\ u(0) = \phi, \end{cases} \tag{1.1}$$

for rough initial data $\phi \in H^s(\mathbb{R})$, $s < 0$. This problem is known [9] to be locally well-posed provided $-\frac{3}{4} < s$. For $s \geq 0$, the local result and L^2 norm conservation imply (1.1) is globally well-posed [1]. Recently, a direct adaptation [7] of Bourgain's high-low frequency technique [3], [2] showed (1.1) is globally well-posed for $\phi \in H^s \cap \dot{H}^a$ for certain $s, a < 0$. A modification of the high-low frequency technique, first used in [8], is presented in this paper which establishes global well-posedness of (1.1) in $H^s(\mathbb{R})$, $-\frac{3}{10} < s$.

A subsequent paper [6] will establish that (1.1) is globally well-posed in $H^s(\mathbb{R})$ for $-\frac{3}{4} < s$. The simplicity of the argument presented here may extend more easily to other situations, such as in our treatment [5] of cubic NLS on $\mathbb{R}^2$ and NLS with derivative in $\mathbb{R}$ [4].

The Multiplier operator I

Let $s < 0$ and $N \gg 1$ be fixed. Define the Fourier multiplier operator

$$\widehat{Iu}(\xi) = m(\xi)\widehat{u}(\xi), \; m(\xi) = \begin{cases} 1, & |\xi| < N, \\ N^{-s}|\xi|^s, & |\xi| \geq 10N \end{cases} \tag{1.2}$$

with m smooth and monotone. The operator I (barely) maps $H^s(\mathbb{R}) \longmapsto L^2(\mathbb{R})$. Observe that on low frequencies $\{\xi : |\xi| < N\}$, I is the identity operator. Note also that I commutes with differential operators. The operator I^{-1} is the Fourier multiplier operator with multiplier $\frac{1}{m(\xi)}$.

An almost L^2 conservation property of (1.1)

1991 *Mathematics Subject Classification.* 35Q53, 42B35, 37K10.

Key words and phrases. Korteweg-de Vries equation, nonlinear dispersive equations, bilinear estimates.

J.E.C. is supported in part by an N.S.F. Postdoctoral Research Fellowship.

M.K. is supported in part by N.S.F. Grant DMS 9801558.

G.S. is supported in part by N.S.F. Grant DMS 9800879 and by a Terman Award.

T.T. is a Clay Prize Fellow and is supported in part by grants from the Packard and Sloan Foundations.

Let $\phi \in H^s(\mathbb{R})$, $-\frac{3}{4} < s < 0$ in (1.1). There is a $\delta = \delta(\|\phi\|_{H^s}) > 0$ such that (1.1) is well-posed for $t \in [0, \delta]$. We observe using the Fundamental Theorem of Calculus, the equation, and integration by parts that

$$\begin{aligned}
\|Iu(\delta)\|_{L^2}^2 &= \|Iu(0)\|_{L^2}^2 + \int_0^\delta \frac{d}{d\tau}(Iu(\tau), Iu(\tau))d\tau, \\
&= \|Iu(0)\|_{L^2}^2 + 2\int_0^\delta (I\dot{u}(\tau), Iu(\tau))d\tau, \\
&= \|Iu(0)\|_{L^2}^2 + 2\int_0^\delta (I(-u_{xxx} - \frac{1}{2}\partial_x[u^2])(\tau), Iu(\tau))d\tau \\
&= \|Iu(0)\|_{L^2}^2 + \int_0^\delta (I(-\partial_x[u^2]), Iu)d\tau.
\end{aligned}$$

Finally, we add $0 = \int_0^\delta \int \partial_x (I(u)^2) I(u) d\tau$ to observe

$$\|Iu(\delta)\|_{L^2}^2 = \|Iu(0)\|_{L^2}^2 + \int_0^\delta \int \partial_x \left\{ (I(u))^2 - I(u^2) \right\} Iu \, dx d\tau. \tag{1.3}$$

This last step enables us to take advantage of some internal cancellation. We apply Cauchy-Schwarz as in [10] and bound the integral above by

$$\left\| \partial_x \{ (I(u))^2 - I(u^2) \} \right\|_{X^\delta_{0,-\frac{1}{2}-}} \|Iu\|_{X^\delta_{0,\frac{1}{2}+}}. \tag{1.4}$$

Remark 1. An effort to find a term providing more cancellation than $\int_0^\delta \int \partial_x (I(u)^2) I(u) d\tau$ used above led to the general procedure described in [6].

Proposition 1. *(A variant of local well-posedness) The initial value problem* (1.1) *is locally well-posed in the Banach space* $I^{-1}L^2 = \{\phi \in H^s$ *with norm* $\|I\phi\|_{L^2}\}$. *with existence lifetime* δ *satisfying*

$$\delta \gtrsim \|I\phi\|_{L^2}^{-\alpha}, \text{ for some } \alpha > 0, \tag{1.5}$$

and moreover

$$\|Iu\|_{X^\delta_{0,\frac{1}{2}+}} \leq C\|I\phi\|_{L^2}. \tag{1.6}$$

This proposition is not difficult to prove using the argument in [9]. Using Duhamel's formula and $X_{s,b}$ space properties reduces matters to proving the bilinear estimate

$$\|\partial_x I(uv)\|_{X_{0,-\frac{1}{2}+}} \leq C\|Iu\|_{X_{0,\frac{1}{2}+}} \|Iv\|_{X_{0,\frac{1}{2}+}} \tag{1.7}$$

to obtain the contraction. The space-time norm bound is then implied by the contraction estimate. The estimate (1.7) follows from the next proposition and the bilinear estimate of Kenig, Ponce and Vega [9].

Proposition 2. *(Extra smoothing) The bilinear estimate*

$$\|\partial_x \{I(u)I(v) - I(uv)\}\|_{X^\delta_{0,-\frac{1}{2}-}} \leq CN^{-\frac{3}{4}+} \|Iu\|_{X^\delta_{0,\frac{1}{2}+}} \|Iv\|_{X^\delta_{0,\frac{1}{2}+}}. \tag{1.8}$$

holds.

Recall the bilinear estimate $\|\partial_x(uv)\|_{X_{0,-\frac{1}{2}+}} \leq C\|u\|_{X_{0,\frac{1}{2}+}}\|v\|_{X_{0,\frac{1}{2}+}}$ from [9]. Proposition 2 reveals a smoothing beyond the recovery of the first derivative for the particular quadratic expression encountered above in (1.3). We prove Proposition 2 in the next section.

The required pieces are now in place for us to give the proof of global well-posedness of (1.1) in $H^s(\mathbb{R})$, $-\frac{3}{10} < s$. Global well-posedness of (1.1) will follow if we show well-posedness on $[0,T]$ for arbitrary $T > 0$. We renormalize things a bit via scaling. If u solves (1.1) then $u_\lambda(x,t) = (\frac{1}{\lambda})^2 u(\frac{x}{\lambda}, \frac{t}{\lambda^3})$ solves (1.1) with initial data $\phi_\lambda(x,t) = (\frac{1}{\lambda})^2 \phi(\frac{x}{\lambda})$. Note that u exists on $[0,T]$ if and only if u_λ exists on $[0,\lambda^3 T]$. A calculation shows that

$$\|I\phi_\lambda\|_{L^2} \leq C\lambda^{-\frac{3}{2}-s} N^{-s} \|\phi\|_{H^s}. \tag{1.9}$$

Here $N = N(T)$ will be selected later but we choose $\lambda = \lambda(N)$ right now by requiring

$$C\lambda^{-\frac{3}{2}-s} N^{-s} \|\phi\|_{H^s} \sim 1 \implies \lambda \sim N^{-\frac{2s}{3+2s}}. \tag{1.10}$$

We now drop the λ subscript on ϕ by assuming that

$$\|I\phi\|_{L^2} = \epsilon_0 \ll 1 \tag{1.11}$$

and our goal is to construct the solution of (1.1) on the time interval $[0,\lambda^3 T]$.

The local well-posedness result of Proposition 1 shows we can construct the solution for $t \in [0,1]$ if we choose ϵ_0 small enough. The almost L^2 conservation property shows $\|Iu(1)\|_2^2 \leq \|Iu(0)\|_2^2 + N^{-\frac{3}{4}+}\|Iu\|^3_{X_{0,\frac{1}{2}+}}$. Using (1.6) and (1.11) gives

$$\|Iu(1)\|_2^2 \leq \epsilon_0^2 + N^{-\frac{3}{4}+}.$$

We can iterate this process $N^{\frac{3}{4}-}$ times before doubling $\|Iu(t)\|_{L^2}$. Therefore, we advance the solution by taking $N^{\frac{3}{4}-}$ time steps of size $O(1)$. We now restrict s by demanding that

$$N^{\frac{3}{4}-} \gtrsim \lambda^3 T = N^{\frac{-6s}{3+2s}} T \tag{1.12}$$

is ensured for large enough N, so $s > -\frac{3}{10}$.

2. Proof of the bilinear smoothing estimate

This section establishes Proposition 2. We distinguish the **very low frequencies** $\{\xi : |\xi| \lesssim 1\}$, the **low frequencies** $\{\xi : 1 \lesssim |\xi| \lesssim \frac{1}{2}N\}$ and the **high frequencies** $\{\xi : \frac{1}{2}N \lesssim |\xi|\}$. Decompose the factor u in the bilinear estimate by writing $u = u_{vl} + u_l + u_h$ with $\widehat{u_l}$ supported on the low frequencies and similarly for the very low and high frequency pieces. We decompose v the same way. Since I is the identity operator on the low and very low frequencies, we can assume one of the factors u, v in the estimate to be shown has its Fourier transform supported in the high frequencies. Symmetry allows us to assume $u = u_h$ and we need to consider the three possible interactions of u_h with v_{vl}, v_l and v_h. Finally, since we are considering (weighted) L^2 norms, we can replace $\widehat{u}$ and $\widehat{v}$ by $|\widehat{u}|$ and $|\widehat{v}|$. Assume therefore that $\widehat{u}, \widehat{v} \geq 0$.

Very low/high interaction

An explicit calculation shows that

$$\mathcal{F}\left(\partial_x\{I(u_h v_{vl}) - I(u_h)v_{vl}\}\right)(\xi) = \int_{\xi=\xi_1+\xi_2} i\xi[m(\xi) - m(\xi_1)]\widehat{u_h}(\xi_1)\widehat{v_{vl}}(\xi_2), \tag{2.1}$$

where $\mathcal{F}$ denotes the Fourier transform. The mean value theorem gives

$$|m(\xi) - m(\xi_1)| \leq |m'(\tilde{\xi_1})||\xi_2|,$$

which may be interpolated with the trivial estimate to give

$$|m(\xi) - m(\xi_1)| \leq CN^{-s}|\xi_1|^s|\xi_1|^{-\theta}|\xi_2|^\theta \tag{2.2}$$

for $0 \leq \theta \leq 1$. Recall that m was defined to be smooth and monotone in (1.2).

Therefore, upon defining $\mathcal{F}(\nabla^\theta f)(\xi) = |\xi|^\theta \widehat{f}(\xi)$, we can write

$$|\mathcal{F}(\partial_x\{I(u_h v_{vl}) - I(u_h)v_{vl}\})(\xi)| \leq |\mathcal{F}(\partial_x(\nabla^{-\theta} I(u_h)(\nabla^\theta v_{vl}))(\xi)|.$$

We now estimate the left side of the bilinear estimate in this interaction by

$$\left\|\partial_x(\nabla^{-\theta} I(u_h))(\nabla^\theta v_{vl})\right\|_{X_{0,\frac{1}{2}+}} \tag{2.3}$$

and by the bilinear estimate of Kenig, Ponce and Vega

$$\leq C\left\|\nabla^{-\theta} I(u_h)\right\|_{X_{0,\frac{1}{2}+}} \left\|\nabla^\theta v_{vl}\right\|_{X_{0,\frac{1}{2}+}}. \tag{2.4}$$

The frequency support of v_{vl} shows that $\left\|\nabla^\theta v_{vl}\right\|_{X_{0,\frac{1}{2}+}} \lesssim \|v_{vl}\|_{X_{0,\frac{1}{2}+}}$. A moments thought shows

$$\left\|\nabla^{-\theta} I(u_h)\right\|_{X_{0,\frac{1}{2}+}} \leq N^{-\theta}\|I(u_h)\|_{X_{0,\frac{1}{2}+}} \tag{2.5}$$

and the claim of the Proposition follows for the (very low)(high) interaction by choosing $\theta > \frac{3}{4}$.

Low/high interaction

The preceding calculations reduce matters to controlling

$$\left\|\partial_x \nabla^{-\theta} I(u_h)\nabla^\theta v_l\right\|_{X_{0,\frac{1}{2}+}} \tag{2.6}$$

and we know that $\widehat{u_h}$ and $\widehat{v_l}$ are supported outside the very low frequencies.

Lemma 1. *Assume $\widehat{u}$ and $\widehat{v}$ are supported outside $\{|\xi| < 1\}$. Then*

$$\|\partial_x(uv)\|_{X_{\alpha,-\frac{1}{2}+}} \leq C\|u\|_{X_{-\gamma_1,\frac{1}{2}+}}\|v\|_{X_{-\gamma_2,\frac{1}{2}+}} \tag{2.7}$$

provided

$$\begin{aligned} \alpha - (\gamma_1 + \gamma_2) &< \frac{3}{4}, \\ \alpha - \gamma_i &< \frac{1}{2},\ i = 1, 2. \end{aligned}$$

We will apply the lemma momentarily with $\alpha = 0, \gamma_1 = \gamma_2 = -\frac{3}{8}+$.

The proof of the lemma is contained in the proof of Theorem 2 in [7]. In particular, the support properties on $\widehat{u}$, $\widehat{v}$ reduce matters to considering Cases A.3, A.4, A.6, B.3, B.4, B.5 and B.6 in [7]. The restriction $\alpha - (\gamma_1 + \gamma_2) < \frac{3}{4}$ arises in Case A.4.c.ii of [7] which is the region containing the counterexample of [9]. Case B.4.b of [7] requires the other condition $\alpha - \gamma_i < \frac{1}{2}$.

The lemma applied to (2.6) gives

$$\leq C\left\|\nabla^{-\theta} I(u_h)\right\|_{X_{-\frac{3}{8}+,\frac{1}{2}+}} \left\|\nabla^{\theta} v_l\right\|_{X_{-\frac{3}{8}+,\frac{1}{2}+}}.$$

Setting $\theta = \frac{3}{8}-$ leaves

$$C\left\|\nabla^{-\frac{3}{4}+} I(u_h)\right\|_{X_{0,\frac{1}{2}+}} \|v_l\|_{X_{0,\frac{1}{2}+}} \leq C N^{-\frac{3}{4}+} \|I(u_h)\|_{X_{0,\frac{1}{2}+}} \|v_l\|_{X_{0,\frac{1}{2}+}}$$

which was to be shown.

High/high interaction

In this region of the interaction, we do not take advantage of any cancellation and estimate the difference with the triangle inequality

$$\|\partial_x\{I(u_h)I(v_h)\}\|_{X_{0,-\frac{1}{2}+}} + \|\partial_x\{I(u_h v_h)\}\|_{X_{0,-\frac{1}{2}+}}.$$

For the first contribution we use the lemma to get

$$\|I(u_h)\|_{X_{-\frac{3}{8}+,\frac{1}{2}+}} \|I(v_h)\|_{X_{-\frac{3}{8}+,\frac{1}{2}+}} \leq N^{-\frac{3}{4}+} \|I(u_h)\|_{X_{0,\frac{1}{2}+}} \|I(v_h)\|_{X_{0,\frac{1}{2}+}}. \tag{2.8}$$

The second contribution is bounded by throwing away I and applying the lemma,

$$\begin{aligned} \|\partial_x\{u_h v_h\}\|_{X_{0,-\frac{1}{2}+}} &\leq \|u_h\|_{X_{-\frac{3}{8}+,\frac{1}{2}+}} \|u_h\|_{X_{-\frac{3}{8}+,\frac{1}{2}+}} \\ &\leq N^{-\frac{3}{8}+s+} \|u_h\|_{X_{s,\frac{1}{2}+}} N^{-\frac{3}{8}+s+} \|v_h\|_{X_{s,\frac{1}{2}+}} \\ &\leq N^{-\frac{3}{4}+} \|u_h\|_{X_{0,\frac{1}{2}+}} \|v_h\|_{X_{0,\frac{1}{2}+}}. \end{aligned}$$

References

[1] J. Bourgain. Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations I,II. *Geom. Funct. Anal.*, 3:107–156, 209–262, 1993.

[2] J. Bourgain. Refinements of Strichartz' inequality and applications to 2D-NLS with critical nonlinearity. *International Mathematical Research Notices*, 5:253–283, 1998.

[3] J. Bourgain. *Global solutions of nonlinear Schrödinger equations.* American Mathematical Society, Providence, RI, 1999.

[4] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Global well-posedness for Schrödinger equations with derivative. (preprint), 2001.

[5] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Global well-posedness of 2d NLS. (in preparation), 2001.

[6] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Sharp Global well-posedness of periodic and nonperiodic generalized Korteweg-de Vries equations. (in preparation), 2001.

[7] J. E. Colliander, G. Staffilani, and H. Takaoka. Global wellposedness of KdV below L^2. *Mathematical Research Letters*, 6(5,6):755–778, 1999.

[8] M. Keel and T. Tao. Local and Global Well-Posedness of Wave Maps on $\mathbb{R}^{1+1}$ for Rough Data. *International Mathematical Research Notices*, 21:1117–1156, 1998.

[9] C. Kenig, G. Ponce, and L. Vega. A bilinear estimate with applications to the KdV equation. *J. Amer. Math. Soc.*, 9:573–603, 1996.

[10] G. Staffilani. On the growth of high Sobolev norms of solutions for KdV and Schrödinger equations. *Duke Math. J.*, 86(1):109–142, 1997.

University of California, Berkeley

Caltech

Stanford University

Hokkaido University

University of California, Los Angeles