GLOBAL WELL-POSEDNESS FOR SCHRÖDINGER EQUATIONS WITH DERIVATIVE

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ABSTRACT. We prove that the 1D Schrödinger equation with derivative in the nonlinear term is globally well-posed in H^s, for $s > 2/3$ for small L^2 data. The result follows from an application of the "I-method". This method allows to define a modification of the energy norm H^1 that is "almost conserved" and can be used to perform an iteration argument. We also remark that the same argument can be used to prove that any quintic nonlinear defocusing Schrödinger equation on the line is globally well-posed for large data in H^s, for $s > 2/3$.

1. INTRODUCTION

We consider the derivative nonlinear Schrödinger initial value problem (IVP)

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\lambda \partial_x(|u|^2 u), \\ u(x,0) = u_0(x), \qquad\qquad x \in \mathbb{R}, t \in \mathbb{R}, \end{cases} \tag{1}$$

where $\lambda \in \mathbb{R}$. The equation in (1) is a model for the propagation of circularly polarized Alfvén waves in magnetized plasma with a constant magnetic field [18, 19, 22].

It is natural to impose the smallness condition

$$\|u_0\|_{L^2} < \sqrt{\frac{2\pi}{\lambda}} \tag{2}$$

on the initial data, as this will force the energy to be positive via the sharp Gagliardo-Nirenberg inequality. Note that the L^2 norm is conserved by the evolution.

Well-posedness for the Cauchy problem (1) has been studied by many authors [10, 11, 12, 20, 21, 25, 26]. The best local well-posedness result is due to Takaoka [21], where a gauge transformation and the Fourier restriction method is used to obtain local well-posedness in H^s, $s \geq 1/2$. In [23] Takaoka showed this result is sharp in the sense that the data map fails to be C^3 or uniformly C^0 for $s < 1/2$ (cf. Bourgain [4] and Biagioni-Linares [1]).

In [20] global well-posedness is obtained for (1) in H^1 assuming the smallness condition (2). The argument there is based on two gauge transformations performed in order to remove the derivative in the nonlinear term. This was improved by Takaoka [23], who proved global well-posed in H^s for $s > \frac{32}{33}$ assuming (2). The method of proof is based on the idea of Bourgain [3, 5] of estimating separately the evolution of low frequencies and of high frequencies of the initial data.

In this paper we improve the global well-posedness result further:

Theorem 1.1. *The Cauchy problem* (1) *is globally well-posed in* H^s *for* $s > 2/3$, *assuming the smallness condition* (2).

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The proof of Theorem 1.1 is based on the "I-method" used by the authors in other nonlinear Cauchy problems in [15], [7], [8], [9] (see also [14]). The basic idea is as follows. After a rescaling, we define a new energy $E_N(u)(t)$ for the solution u that depends on a parameter $N \gg 1$. We prove a local well-posed result in the norm associated to E_N on intervals of length ~ 1, and finally we perform an iteration on the time intervals. The reason why this iteration can be globally extended is that the increment of the energy $E_N(u)(t)$ over each time interval is very small. In other words the argument is successful because the energy $E_N(u)(t)$ is *almost conserved.*

After the proof of Theorem 1.1 is completed, we will briefly remark that using the same techniques one can also show that the 1D defocusing quintic nonlinear Schrödinger is global well-posed for initial data in $H^s, s > 2/3$. The details of the proof of this fact will appear in a different paper.

The restriction $s > 2/3$ is probably not sharp, and might be improvable either by more sophisticated multilinear estimates and better estimates on the symbols M_4, M_6, M_8 which appear in our argument, or by using the "correction term" strategy of [8]. In fact one may reasonably conjecture that one could extend the global well-posedness result to match the local result at $s > 1/2$. We will not pursue these matters here.

2. Notation

To prove Theorem 1.1 we may assume $2/3 < s < 1$, since for the $s \geq 1$ the result is contained in [20, 23]. Henceforth $2/3 < s < 1$ shall be fixed. Also, by rescaling u, we may assume $\lambda = 1$.

We use C to denote various constants depending on s; if C depends on other quantities as well, this will be indicated by explicit subscripting, e.g. $C_{\|u_0\|_2}$ will depend on both s and $\|u_0\|_2$. We use $A \lesssim B$ to denote an estimate of the form $A \leq CB$. We use $a+$ and $a-$ to denote expressions of the form $a + \varepsilon$ and $a - \varepsilon$, where $0 < \varepsilon \ll 1$ depends only on s.

We use $\|f\|_p$ to denote the $L^p(\mathbb{R})$ norm, and $L^q_t L^r_x$ to denote the mixed norm

$$\|f\|_{L^q_t L^r_x} := (\int \|f(t)\|_r^q \ dt)^{1/q}$$

with the usual modifications when $q = \infty$.

We define the spatial Fourier transform of $f(x)$ by

$$\hat{f}(\xi) := \int_{\mathbb{R}} e^{-ix\xi} f(x) \ dx$$

and the spacetime Fourier transform $u(t, x)$ by

$$\tilde{u}(\tau, \xi) := \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(x\xi + t\tau)} u(t, x) \ dtdx.$$

Note that the derivative ∂_x is conjugated to multiplication by $i\xi$ by the Fourier transform.

We shall also define D_x to be the Fourier multiplier with symbol $\langle \xi \rangle := 1 + |\xi|$. We can then define the Sobolev norms H^s by

$$\|f\|_{H^s} := \|D_x^s f\|_2 = \|\langle \xi \rangle^s \hat{f}\|_{L^2_\xi}.$$

We also define the spaces $X^{s,b}(\mathbb{R} \times \mathbb{R})$ (first introduced in [2]) on $\mathbb{R} \times \mathbb{R}$ by

$$\|u\|_{X^{s,b}(\mathbb{R} \times \mathbb{R})} := \|\langle \xi \rangle^s \langle \tau - |\xi|^2 \rangle^b \hat{u}(\xi, \tau)\|_{L^2_\tau L^2_\xi}.$$

We often abbreviate $\|u\|_{s,b}$ for $\|u\|_{X^{s,b}(\mathbb{R} \times \mathbb{R})}$. For any time interval I, we define the restricted spaces $X^{s,b}(I \times \mathbb{R})$ by

$$\|u\|_{X^{s,b}(I \times \mathbb{R})} := \inf\{\|U\|_{s,b} : U|_{I \times \mathbb{R}} = u\}.$$

We shall take advantage of the Strichartz estimates

$$\|u\|_{L_t^6 L_x^6} \lesssim \|u\|_{0,1/2+} \tag{3}$$

and

$$\|u\|_{L_t^\infty L_x^2} \lesssim \|u\|_{0,1/2+} \tag{4}$$

(see e.g. [2]). From (4) and Sobolev embedding we observe

$$\|u\|_{L_t^\infty L_x^\infty} \lesssim \|u\|_{1/2+,1/2+} \tag{5}$$

In our arguments we shall be using the trivial embedding

$$\|u\|_{s_1,b_1} \lesssim \|u\|_{s_2,b_2} \text{ whenever } s_1 \leq s_2, b_1 \leq b_2$$

so frequently that we will not mention this embedding explicitly.

We now give some useful notation for multilinear expressions. If $n \geq 2$ is an even integer, we define a *(spatial) multiplier of order* n to be any function $M_n(\xi_1, \ldots, \xi_n)$ on the hyperplane

$$\Gamma_n := \{(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n : \xi_1 + \ldots + \xi_n = 0\},$$

which we endow with the standard measure $\delta(\xi_1 + \ldots + \xi_n)$, where δ is the Dirac delta.

If M_n is a multiplier of order n and $f_1, \ldots, f_n$ are functions on $\mathbb{R}$, we define the quantity $\Lambda_n(M_n; f_1, \ldots, f_n)$ by

$$\Lambda_n(M_n; f_1, \ldots, f_n) := \int_{\Gamma_n} M_n(\xi_1, \ldots, \xi_n) \prod_{j=1}^n \hat{f}_j(\xi_j).$$

We adopt the notation

$$\Lambda_n(M_n; f) := \Lambda_n(M_n; f, \bar{f}, f, \bar{f}, \ldots, f, \bar{f}).$$

Observe that $\Lambda_n(M_n; f)$ is invariant under permutations of the even ξ_j indices, or of the odd ξ_j indices.

If M_n is a multiplier of order n, $1 \leq j \leq n$ is an index, and $k \geq 1$ is an even integer, we define the *elongation* $\mathbf{X}_j^k(M_n)$ of M_n to be the multiplier of order $n+k$ given by

$$\mathbf{X}_j^k(M_n)(\xi_1, \ldots, \xi_{n+k}) := M_n(\xi_1, \ldots, \xi_{j-1}, \xi_j + \ldots + \xi_{j+k}, \xi_{j+k+1}, \ldots, \xi_{n+k}).$$

In other words, $\mathbf{X}_j^k$ is the multiplier obtained by replacing ξ_j by $\xi_j + \ldots + \xi_{j+k}$ and advancing all the indices after ξ_j accordingly.

We shall often write ξ_{ij} for $\xi_i + \xi_j$, ξ_{ijk} for $\xi_i + \xi_j + \xi_k$, etc. We also write ξ_{i-j} for $\xi_i - \xi_j$, ξ_{ij-klm} for $\xi_{ij} - \xi_{klm}$, etc.

3. The Gauge Transformation and the conservation laws

In this section we apply the gauge transform used in [20] in order to improve the derivative nonlinearity.

Definition 3.1. We define the non-linear map $\mathcal{G} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$\mathcal{G}f(x) := e^{-i\int_{-\infty}^x |f(y)|^2 dy} f(x).$$

The inverse transform $\mathcal{G}^{-1}f$ is then given by

$$\mathcal{G}^{-1}f(x) := e^{i\int_{-\infty}^x |f(y)|^2 dy} f(x).$$

This transform is well behaved on H^s:

Lemma 3.2. *The map $\mathcal{G}$ is a bicontinuous map from H^s to H^s.*

A similar statement holds for $0 \leq s \leq 1/2$, but we shall not need it here.

Proof. We shall just prove the continuity of $\mathcal{G}$, as the continuity of $\mathcal{G}^{-1}$ is proven similarly.

Define Lip to be the space of functions with norm

$$\|f\|_{Lip} := \|f\|_\infty + \|f'\|_{L^\infty}.$$

Since $s > 1/2$, we see from Sobolev embedding that the nonlinear map $f \mapsto e^{-i\int_{-\infty}^x |f(y)|^2 dy}$ continuously maps H^s to Lip. It therefore suffices to show the product estimate

$$\|fg\|_{H^s} \lesssim \|f\|_{H^s}\|g\|_{Lip}.$$

But this estimate follows immediately from the Leibnitz rule and Hölder when $s = 0$ or $s = 1$, and the intermediate cases then follow by interpolation. □

Set $w_0 := \mathcal{G}u_0$, and $w(t) := \mathcal{G}u(t)$ for all times t. A straightforward calculation shows that the IVP (1) can be transformed to

$$\begin{cases} i\partial_t w + \partial_x^2 w = -iw^2\partial_x \bar{w} - \frac{1}{2}|w|^4 w, \\ w(x,0) = w_0(x), \qquad x \in \mathbb{R}, t \in \mathbb{R}, \end{cases} \tag{6}$$

Also, the smallness condition (2) becomes

$$\|w_0\|_{L^2} < \sqrt{2\pi}. \tag{7}$$

By Lemma 3.2 we thus see that global well-posedness of (1) in H^s is equivalent to that of (6). From [20, 21, 23], we know that both Cauchy problems are locally well-posed in H^s and globally well-posed in H^1 assuming (7). By standard limiting arguments, we thus see that Theorem 1.1 will follow if we can show

Proposition 3.3. *Let w be a global H^1 solution to (6) obeying (7). Then for any $T > 0$ we have*

$$\sup_{0 \leq t \leq T} \|w(t)\|_{H^s} \lesssim C_{\|w_0\|_{H^s}, \|w_0\|_2, T}$$

where the right-hand side does not depend on the H^1 norm of w.

Just by looking at the equation in (6) it is not easy to understand why this should be better than the equation in (1). In fact we still see a derivative, and moreover a quintic nonlinearity has been introduced. But it was made clear in [20, 13, 21] how a derivative of the complex conjugate of the solution w can be handled while a derivative of w cannot. Also the quintic term is not going to introduce any extra trouble.

Let $n \geq 2$ be an even integer, and let M_n be a multiplier of order n. From (6) we have

$$\partial_t w = iw_{xx} - w\bar{w}_x w + \frac{i}{2} w\bar{w}w\bar{w}w$$

and

$$\partial_t \bar{w} = -iw_{xx} - \bar{w}w_x\bar{w} - \frac{i}{2}\bar{w}w\bar{w}w\bar{w}.$$

Taking the Fourier transform of these identities, we obtain the useful differentiation law

$$\begin{aligned} \partial_t \Lambda_n(M_n; w(t)) &= i\Lambda_n(M_n \sum_{j=1}^n (-1)^j \xi_j^2; w(t)) \\ &\quad - i\Lambda_{n+2}(\sum_{j=1}^n \mathbf{X}_j^2(M_n)\xi_{j+1}; w(t)) \\ &\quad + \frac{i}{2}\Lambda_{n+4}(\sum_{j=1}^n (-1)^{j-1}\mathbf{X}_j^4(M_n); w(t)) \end{aligned} \tag{8}$$

for any even integer $n \geq 2$ and any multiplier M_n of order n.

We now turn to the conservation laws that the solution w of (6) enjoys. What follows in this section was originally described by Ozawa in [20], however we have redone the computations in our own notation as this will prove useful later.

Definition 3.4. If $f \in H^1(\mathbb{R})$, we define the energy $E(f)$ by

$$E(f) := \int \partial_x f \partial_x \overline{f}\, dx - \frac{1}{2}\mathrm{Im} \int f\overline{f} f \partial_x \overline{f}\, dx.$$

By Plancherel, we may write $E(f)$ using the Λ notation as

$$E(f) = -\Lambda_2(\xi_1\xi_2; f) - \frac{1}{2}\mathrm{Im}\Lambda_4(i\xi_4; f).$$

Expanding out the second term using $\mathrm{Im}(z) = (z - \bar{z})/2i$, and using symmetry, we may rewrite this as

$$E(f) = -\Lambda_2(\xi_1\xi_2; f) + \frac{1}{8}\Lambda_4(\xi_{13-24}; f). \tag{9}$$

Lemma 3.5. [20] *If w is an H^1 solution to (6) for times $t \in [0, T]$, then we have*

$$\|w(t)\|_2 = \|w_0\|_2$$

and

$$E(w(t)) = E(w_0)$$

for all $t \in [0, T]$.

Proof. These conservation laws are proven in [20], however we give a proof based on the identity (8), as the proof here will be needed later on.

We of course have

$$\|w(t)\|_2^2 = \Lambda_2(1; w(t)).$$

In the rest of this proof we shall drop the $w(t)$ from the Λ notation. Differentiating the previous and applying (8), we obtain

$$\partial_t \|w(t)\|_2^2 = -i\Lambda_2(\xi_1^2 - \xi_2^2) - i\Lambda_4(\xi_2 + \xi_3) + \frac{i}{2}\Lambda_6(1 - 1 + 1 - 1 + 1 - 1).$$

The first term vanishes since $\xi_{12} = 0$. The second term can be symmetrized to $-\frac{i}{2}\Lambda_4(\xi_{1234})$ which vanishes. The third term clearly vanishes. This proves the L^2 conservation.

Now we prove energy conservation. From (9) we have

$$\partial_t E(t) = -\partial_t \Lambda_2(\xi_1\xi_2) + \frac{1}{8}\partial_t \Lambda_4(\xi_{13-24}) \tag{10}$$

and from (8)

$$\partial_t \Lambda_2(\xi_1\xi_2) = -i\Lambda_2(\xi_1\xi_2(\xi_1^2 - \xi_2^2)) - i\Lambda_4(\xi_{123}\xi_4\xi_2 + \xi_1\xi_{234}\xi_3) + \frac{i}{2}\Lambda_6(\xi_{12345}\xi_6 - \xi_1\xi_{23456}).$$

The Λ_2 term vanishes since $\xi_{12} = 0$. To simplify the Λ_4 term we write $\xi_{123} = -\xi_4$, $\xi_{234} = -\xi_1$ and then symmetrize. To simplify the Λ_6 term we write $\xi_{12345} = -\xi_6$, $\xi_{23456} = -\xi_1$ and then symmetrize, to obtain

$$\partial_t \Lambda_2(\xi_1\xi_2) = \frac{i}{2}\Lambda_4(\xi_1^2\xi_3 + \xi_2^2\xi_4 + \xi_3^2\xi_1 + \xi_4^2\xi_2) + \frac{i}{6}\Lambda_6(\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2 + \xi_5^2 - \xi_6^2).$$

We may simplify the Λ_4 term further, using the identity

$$\begin{aligned}\xi_1^2\xi_3 + \xi_2^2\xi_4 + \xi_3^2\xi_1 + \xi_4^2\xi_2 &= \xi_1\xi_3\xi_{13} + \xi_2\xi_4\xi_{24} \\ &= \xi_{13}(\xi_1\xi_3 - \xi_2\xi_4) \\ &= \xi_{13}(-\xi_1\xi_{124} - \xi_2\xi_4) \\ &= -\xi_{13}(\xi_1+\xi_2)(\xi_1+\xi_4) \\ &= -\xi_{12}\xi_{13}\xi_{14}\end{aligned}$$

to obtain

$$\partial_t \Lambda_2(\xi_1\xi_2) = -\frac{i}{2}\Lambda_4(\xi_{12}\xi_{13}\xi_{14}) + \frac{i}{6}\Lambda_6(\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2 + \xi_5^2 - \xi_6^2). \tag{11}$$

We now consider the second component of the energy. From (8) we have

$$\begin{aligned}\partial_t \Lambda_4(\xi_{13-24}) &= i\Lambda_4(\xi_{13-24}(\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2)) \\ &\quad - i\Lambda_6(\xi_{1235-46}\xi_2 + \xi_{15-2346}\xi_3 + \xi_{1345-26}\xi_4 + \xi_{13-2456}\xi_5) \\ &\quad + \frac{i}{2}\Lambda_8(\xi_{123457-68} - \xi_{17-234568} + \xi_{134567-28} - \xi_{13-245678}).\end{aligned}$$

The Λ_8 term symmetrizes to $i\Lambda_8(\xi_{12345678})$ which vanishes. The Λ_6 term can be rewritten as

$$2i\Lambda_6(\xi_{46}\xi_2 - \xi_{15}\xi_3 + \xi_{26}\xi_4 - \xi_{13}\xi_5)$$

which we rewrite as

$$2i\Lambda_6(\xi_{246}\xi_2 - \xi_{135}\xi_3 + \xi_{246}\xi_4 - \xi_{135}\xi_5) - 2i\Lambda_6(\xi_2^2 - \xi_3^2 + \xi_4^2 - \xi_5^2).$$

The first term symmetrizes to $\frac{4i}{3}\Lambda_6(\xi_{246}^2 - \xi_{135}^2)$ which vanishes. The second term symmetrizes to

$$\frac{4i}{3}\Lambda_6(\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2 + \xi_5^2 - \xi_6^2).$$

Finally, consider the Λ_4 term. We may factorize

$$\xi_{13-24}(\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2) = \xi_{13-24}(\xi_{1-2}\xi_{12} + \xi_{3-4}\xi_{34}).$$

Since $\xi_{12} = \xi_{-34}$ and $\xi_{13} = -\xi_{24}$, we may simplify this as

$$2\xi_{13}\xi_{12}(\xi_{1-2} - \xi_{3-4}) = 4\xi_{12}\xi_{13}\xi_{14}.$$

Combining all these identities we thus have

$$\frac{1}{8}\partial_t \Lambda_4(\xi_{13-24}) = -\frac{1}{2}i\Lambda_4(\xi_{12}\xi_{13}\xi_{14}) - \frac{i}{6}\Lambda_6(\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2 + \xi_5^2 - \xi_6^2).$$

Combining this with (11) and (10) we obtain

$$\partial_t E(w(t)) = 0$$

and the claim follows. $\square$

Heuristically, the energy $E(w(t))$ has the same strength as $\|w(t)\|_{H^1}^2$. We can make this precise:

Lemma 3.6. *Let f be an H^1 function on $\mathbb{R}$ such that $\|f\|_2 < \sqrt{2\pi}$. Then we have*

$$\|\partial_x f\|_2 \le C_{\|f\|_2} E(f)^{1/2} \tag{12}$$

where $C_{\|f\|_2}$ depends only on $\|f\|_2$.

Proof. Define the function

$$g(x) := \exp(i\frac{3}{4}\int_{-\infty}^{x} |f(y)|^2\, dy)f(x).$$

A routine computation shows that

$$\|g\|_2 = \|f\|_2 < \sqrt{2\pi}$$

and

$$E(f) = \|\partial_x g\|_2^2 - \frac{1}{16}\|g\|_6^6.$$

From the sharp Gagliardo-Nirenberg inequality [27]

$$\|g\|_6^6 \leq \frac{4}{\pi^2}\|g\|_2^4\|\partial_x g\|_2^2 \tag{13}$$

we therefore have

$$\|\partial_x g\|_2 \lesssim C_{\|f\|_2} E(f)^{1/2}.$$

From the definition of g we have

$$f(x) = \exp(-i\frac{3}{4}\int_{-\infty}^{x} |g(y)|^2\, dy)g(x)$$

and so we have

$$\|\partial_x f\|_2 \lesssim \|\partial_x g\|_2 + \|g^3\|_2.$$

By another application of (13) we thus obtain (12). □

4. The Almost Conserved Energy Norm

It remains to prove Proposition 3.3. Fix w, T. We also let $N \gg 1$ be a large parameter depending on T, $\|w_0\|_2$, and $\|w_0\|_{H^s}$ which we shall choose later.

Because we do not want to use the H^1 norm of w, we cannot directly use the energy $E(w(t))$ defined above. So we are looking for a substitute notion of "energy" that can be defined for a less regular solution and that has a very slow increment in time. In the frequency space let us consider an even C^∞ monotone multiplier $m(\xi)$ taking values in $[0,1]$ such that

$$m(\xi) := \begin{cases} 1, & \text{if } |\xi| < N, \\ \left(\frac{|\xi|}{N}\right)^{s-1} & \text{if } |\xi| > 2N. \end{cases} \tag{14}$$

We define the multiplier operator $I : H^s \longrightarrow H^1$ such that $\widehat{Iw}(\xi) := m(\xi)\widehat{w}(\xi)$. This operator is smoothing of order $1-s$; indeed we have

$$\|u\|_{s_0,b_0} \lesssim \|Iu\|_{s_0+1-s,b_0} \lesssim N^{1-s}\|u\|_{s_0,b_0} \tag{15}$$

for any $s_0, b_0 \in \mathbb{R}$.

Our substitute energy will be defined by

$$E_N(w) := E(Iw).$$

Note that this energy makes sense even if w is only in H^s.

In general the energy $E_N(w(t))$ is not conserved in time, but we will show that the increment is very small in terms of N. This will be accomplished in three stages. First, in Proposition 4.1 below, we write the increment of $E_N(w(t))$ as a multilinear expression in w. Then, in Lemma 6.1, we estimate these multilinear expressions in terms of the norm $\|Iw\|_{1,1/2+}$, gaining a power of N^{-1+} in the process. Finally, in Theorem 5.1 (and Lemma 3.6), we control the norm $\|Iw\|_{1,1/2+}$ back in terms of $E_N(w(t))$.

Proposition 4.1. *Let w be an H^1 global solution to (6). Then for any $T \in \mathbb{R}$ and $\delta > 0$ we have*

$$E_N(w(T+\delta)) - E_N(w(T)) = \int_T^{T+\delta} [\Lambda_4(M_4; w(t)) + \Lambda_6(M_6; w(t)) + \Lambda_8(M_8; w(t))] \, dt$$

where the multipliers M_4, M_6, M_8 are given by

$$M_4 := C_1 m_1 m_2 m_3 m_4 \xi_{12}\xi_{13}\xi_{14} + C_2(m_1^2\xi_1^2\xi_3 + m_2^2\xi_2^2\xi_4 + m_3^2\xi_3^2\xi_1 + m_4^2\xi_4^2\xi_2)$$

$$M_6 := C_3 \sum_{j=1}^{6} (-1)^{j-1} m_j^2 \xi_j^2 + C_4 \sum_{\{a,c,e\}=\{1,3,5\},\{b,d,f\}=\{2,4,6\}} m_a m_b m_c m_{def} \xi_{ac}\xi_e - m_{abc} m_d m_e m_f \xi_{df} \xi_b$$

$$M_8 := C_5 \sum_{\{a,c,e,g\}=\{1,3,5,7\};\{b,d,f,h\}=\{2,4,6,8\}} m_a m_b m_c m_{defgh} \xi_{ac-bdefgh} - m_{abcde} m_f m_g m_h \xi_{abcdeg-fh}$$

where $C_1, \ldots, C_5$ are absolute constants and we adopt the abbreviations m_i for $m(\xi_i)$, m_{ij} for $m(\xi_{ij})$, etc. Furthermore, if $|\xi_j| \ll N$ for all j, then the multipliers M_4, M_6, M_8 all vanish.

Proof. From (9) we have

$$E_N(w(t)) = -\Lambda_2(m_1\xi_1 m_2\xi_2; w(t)) + \frac{1}{8}\Lambda_4(\xi_{13-24} m_1 m_2 m_3 m_4; w(t)).$$

Henceforth we omit the $w(t)$ from the Λ notation. By (8) we have

$$\begin{aligned}
\partial_t \Lambda_2(m_1\xi_1 m_2\xi_2) &= -i\Lambda_2(m_1\xi_1 m_2 \xi_2(\xi_1^2 - \xi_2^2)) \\
&\quad - i\Lambda_4(m_{123}\xi_{123} m_4 \xi_4 \xi_2 + m_1\xi_1 m_{234}\xi_{234}\xi_3) \\
&\quad + \frac{i}{2}\Lambda_6(m_{12345}\xi_{12345} m_6 \xi_6 - m_1 \xi_1 m_{23456}\xi_{23456}).
\end{aligned}$$

The Λ_2 term vanishes since $\xi_{12} = 0$. To simplify the Λ_4 term, we use $\xi_{123} = -\xi_4$ and $\xi_{234} = -\xi_1$ and then symmetrize to obtain the second term of M_4. To simplify the Λ_6 term, we use $\xi_{12345} = -\xi_6$ and $\xi_{23456} = -\xi_1$ and then symmetrize to get the first term of M_6.

In a similar vein we have

$$\begin{aligned}
\partial_t \Lambda_4(\xi_{13-24} m_1 m_2 m_3 m_4) &= -i\Lambda_4(\xi_{13-24} m_1 m_2 m_3 m_4 (\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2)) \\
&\quad - i\Lambda_6(\xi_{1235-46} m_{123} m_4 m_5 m_6 \xi_2 + \xi_{15-2346} m_1 m_{234} m_5 m_6 \xi_3 \\
&\qquad + \xi_{1345-26} m_1 m_2 m_{345} m_6 \xi_4 + \xi_{13-2456} m_1 m_2 m_3 m_{456} \xi_5) \\
&\quad + \frac{i}{2}\Lambda_8(\xi_{123457-68} m_{12345} m_6 m_7 m_8 - \xi_{17-234568} m_1 m_{23456} m_7 m_8 \\
&\qquad + \xi_{134567-28} m_1 m_2 m_{34567} m_8 - \xi_{13-245678} m_1 m_2 m_3 m_{45678}).
\end{aligned}$$

The Λ_4 term is of the form of the first term of M_4, by the argument used to prove (11). To simplify the Λ_6 term, we use $\xi_{1235-46} = -2\xi_{46}$ and similarly for the other four terms, then symmetrize to obtain the second term of M_6. Finally if we symmetrize the Λ_8 term we obtain M_8. The first part of the Proposition then follows from the Fundamental Theorem of Calculus applied to the function $t \longrightarrow E_N(w(t))$.

If all the frequencies are $\ll N$, then all the m_i, m_{ij}, etc. terms are equal to 1. In this case our calculations are identical to those in Lemma 3.5 and so our symbols M_4, M_6, M_8 will vanish by the computations given in that Lemma. $\square$

5. Local estimates

In Lemma 6.1 we shall estimate the expression in Proposition 4.1. It turns out that one cannot estimate this expression effectively just by using spatial norms such as $\|Iw\|_{H^1}$ (as is done for some simple equations in [5]), but one must use spacetime norms such as $\|Iw\|_{1,1/2+}$. The purpose of this section is to obtain the required control on these spacetime norms:

Theorem 5.1. *Let w be a H^1 global solution to (6), and let $T \in \mathbb{R}$ be such that*

$$\|Iw(T)\|_{H^1} \leq C_0$$

for some $C_0 > 0$. Then we have

$$\|Iw\|_{X^{1,1/2+}([T,T+\delta]\times\mathbb{R})} \lesssim 1$$

for some $\delta > 0$ depending on C_0.

We now prove Theorem 5.1. We shall be able to exploit the estimates in [21]. By standard iteration arguments (see e.g. [2], [16], [17], [21], [23]) it suffices to prove

Lemma 5.2. *We have*

$$\|I(w_1\partial_x\overline{w_2}w_3)\|_{X_{1,b-1}(\mathbb{R}\times\mathbb{R})} \lesssim \prod_{i=1}^{3}\|Iw_i\|_{X_{1,1/2+}(\mathbb{R}\times\mathbb{R})} \tag{16}$$

$$\|I(w_1\overline{w_2}w_3\overline{w_4}w_5)\|_{X_{1,b-1}(\mathbb{R}\times\mathbb{R})} \lesssim \prod_{i=1}^{5}\|Iw_i\|_{X_{1,1/2+}(\mathbb{R}\times\mathbb{R})}. \tag{17}$$

for all Schwarz functions w_i and some $b > 1/2$ (in fact we may take any $1/2 < b < 5/8$).

Proof. By Plancherel and duality it suffices to show

$$|\int_* \frac{m(\xi_4)\langle\xi_4\rangle\langle\tau_4+\xi_4^2\rangle^{b-1}\xi_2}{\prod_{j=1}^3 m(\xi_j)\langle\xi_j\rangle\langle\tau_j-(-1)^{j-1}\xi_j^2\rangle^{1/2+}}\prod_{j=1}^4 F_j(\tau_j,\xi_j)| \lesssim \prod_{j=1}^4\|F_j\|_{L^2_{\tau_j}L^2_{\xi_j}}$$

and

$$|\int_{**} \frac{m(\xi_6)\langle\xi_6\rangle\langle\tau_6+\xi_6^2\rangle^{b-1}}{\prod_{j=1}^5 m(\xi_j)\langle\xi_j\rangle\langle\tau_j-(-1)^{j-1}\xi_j^2\rangle^{1/2+}}\prod_{j=1}^6 F_j(\tau_j,\xi_j)| \lesssim \prod_{j=1}^6\|F_j\|_{L^2_{\tau_j}L^2_{\xi_j}}$$

for all functions $F_1,\ldots,F_6$, where $\int_*$, $\int_{**}$ denotes integration over the measure $\delta(\tau_1+\ldots+\tau_4)\delta(\xi_1+\ldots+\xi_4)$ and $\delta(\tau_1+\ldots+\tau_6)\delta(\xi_1+\ldots+\xi_6)$ respectively.

We may assume that the F_j are all real and non-negative. We now observe the pointwise estimate

$$\frac{m(\xi_n)\langle\xi_n\rangle^{1-s}}{\prod_{j=1}^{n-1} m(\xi_j)\langle\xi_j\rangle^{1-s}} \lesssim 1$$

for $n = 4, 6$ and all $\xi_1,\ldots,\xi_n$ such that $\xi_1+\ldots+\xi_n = 0$. To see this, we use symmetry to assume that $|\xi_1| \geq \ldots \geq |\xi_{n-1}|$, so that $|\xi_n| \lesssim |\xi_1|$. Since $m(\xi)\langle\xi\rangle^{1-s}$ is essentially increasing in $|\xi|$, we thus see that

$$\frac{m(\xi_n)\langle\xi_n\rangle^{1-s}}{\prod_{j=1}^{n-1} m(\xi_j)\langle\xi_j\rangle^{1-s}} \lesssim \frac{1}{\prod_{j=2}^{n-1} m(\xi_j)\langle\xi_j\rangle^{1-s}}.$$

Since $m(\xi)\langle\xi\rangle^{1-s} \gtrsim 1$ for all ξ, the claim follows.

Because of this estimate, we only need to show the estimates

$$|\int_{*} \frac{\langle\xi_4\rangle^s\langle\tau_4+\xi_4^2\rangle^{b-1}\xi_2}{\prod_{j=1}^3\langle\xi_j\rangle^s\langle\tau_j-(-1)^{j-1}\xi_j^2\rangle^{1/2+}}\prod_{j=1}^4 F_j(\tau_j,\xi_j)| \lesssim \prod_{j=1}^4 \|F_j\|_{L^2_{\tau_j}L^2_{\xi_j}} \tag{18}$$

and

$$|\int_{**} \frac{\langle\xi_6\rangle^s\langle\tau_6+\xi_6^2\rangle^{b-1}}{\prod_{j=1}^5\langle\xi_j\rangle^s\langle\tau_j-(-1)^{j-1}\xi_j^2\rangle^{1/2+}}\prod_{j=1}^6 F_j(\tau_j,\xi_j)| \lesssim \prod_{j=1}^6 \|F_j\|_{L^2_{\tau_j}L^2_{\xi_j}}. \tag{19}$$

The estimate (18) is equivalent to the first estimate of Lemma 3.1 in [23] after undoing the duality and Plancherel, so it suffices to prove (19). By undoing the duality we can write this as

$$\|w_1\overline{w_2}w_3\overline{w_4}w_5\|_{s,b-1} \lesssim \prod_{j=1}^5 \|w_j\|_{s,1/2+}.$$

We may assume that the Fourier transforms $\tilde{w}_j$ are all real and non-negative. By using $|\xi_1+\xi_2+\xi_3+\xi_4+\xi_5|^s \lesssim \sum_{i=1}^5 |\xi_i|^s$, it suffices to prove estimates of the form

$$\|(D_x^s w_1)\overline{w_2}w_3\overline{w_4}w_5\|_{0,b-1} \lesssim \prod_{j=1}^5 \|w_j\|_{s,1/2+},$$

plus similar estimates when D_x^s falls on one of the other functions. We shall only prove the displayed estimate, as the others are similar. We may estimate the $X^{0,b-1}$ norm by the $L^2_tL^2_x$ norm. But then the claim follows from three applications of (3) and two applications of (5), and Hölder (ensuring that the term with the D_x^s is estimated using (3)). $\square$

6. Proof of Proposition 3.3

We can now prove Proposition 3.3, which as remarked before will give Theorem 1.1. Let T, w be as in the Proposition. Our constants may depend on $\|w_0\|_2$ and $\|w_0\|_{H^s}$.

We start by rescaling the solution w. Let $\mu > 0$ be chosen later. We observe that w is a solution for the IVP (6) if and only if

$$w^\mu(t,x) = \frac{1}{\mu^{1/2}} w\left(\frac{t}{\mu^2}, \frac{x}{\mu}\right)$$

is a solution for the IVP (6) with initial data $w_0^\mu = \mu^{-1/2}w(\mu^{-1}x)$. From Plancherel's theorem and a simple computation we see that

$$\|I\partial_x w_0^\mu\|_2 \lesssim \frac{N^{1-s}}{\mu^s}\|w_0\|_{H^s}.$$

while

$$\|Iw_0^\mu\|_2 \le \|w_0^\mu\|_2 = \|w_0\|_2 < \sqrt{2\pi}.$$

We now choose $\mu := N^{\frac{1-s}{s}}$. From the previous we see that $\|Iw_0^\mu\|_{H^1} \lesssim 1$, so from Sobolev embedding (or Gagliardo-Nirenberg) we obtain

$$E(Iw_0^\mu) \le C_1$$

for some constant $C_1 > 0$.

Now suppose inductively that we have a time T such that

$$E(Iw^\mu(T)) \le C_1 + C_2N^{-1+}T$$

where $C_2 > 0$ is a constant depending on C_1 to be chosen later. If $T \ll N^{1-}$, we then have $E(Iw^\mu(T)) \leq 2C_1$, which implies from Lemma 3.6 that

$$\|Iw^\mu(T)\|_{H^1} \leq C_3$$

where C_3 depends on C_1. By Theorem 5.1 we thus have

$$\|Iw^\mu\|_{X^{1,1/2+}([T,T+\delta]\times\mathbb{R})} \leq C_4$$

where C_4, δ depend on C_3.

In the next four sections we shall prove the key estimate

Lemma 6.1. *For any Schwartz function w, we have*

$$|\int_T^{T+\delta} \Lambda_n(M_n; w(t))\, dt| \lesssim N^{-1+} \|Iw\|^n_{X^{1,1/2+}([T,T+\delta]\times\mathbb{R})} \tag{20}$$

for $n = 4, 6, 8$, where M_4, M_6, M_8 are defined in Proposition 4.1.

Assuming this estimate for the moment, we see from the previous and Proposition 4.1 that

$$E(Iw^\mu(T+\delta)) \leq E(Iw^\mu(T)) + C_5 N^{-1+}$$

where C_5 depends on δ and C_4. This allows us to close the induction hypothesis by setting $C_2 := C_5$. As a consequence we have thus shown that[1]

$$\|Iw^\mu(T)\|_{H^1} \lesssim 1$$

for all $T \ll N^{1-}$. From the definition of I this implies that

$$\|w^\mu(T)\|_{H^s} \lesssim C_N$$

for all $T \ll N^{1-}$. Undoing the scaling, this implies that

$$\|w(T)\|_{H^s} \lesssim C_{N,\mu}$$

for all $T \ll N^{1-}/\mu^2$. However, if $s > 2/3$, then $N^{1-}/\mu^2 = N^{\frac{3s-2-}{s}}$ goes to infinity as $N \to \infty$, and Proposition 3.3 follows. $\square$

Remark 6.2. An examination of the above argument shows also that the H^s norm of w (and of u) grows at most polynomially in time, however the order of this growth obtained by this argument goes to infinity as $s \to 2/3$.

7. Proof of Lemma 6.1: preliminaries

To prove Lemma 6.1 we shall treat the cases $n = 4$, $n = 6$, $n = 8$ separately. The idea will be first to obtain some good estimates on M_n in terms of $m(\xi_i)$ and $\langle \xi_i \rangle$, and then to bound the resulting multilinear expression using standard tools such as the Strichartz estimates (3), (5), the trivial estimate

$$\|u\|_{L^2_t L^2_x} \lesssim \|u\|_{0,0}, \tag{21}$$

and Hölder's inequality. In addition to the above linear estimates, we shall also take advantage of the following bilinear improvement to Strichartz' estimate in the case of differing frequencies (cf. [3])

Lemma 7.1. *For any Schwartz functions u, v with Fourier support in $|\xi| \sim R$, $|\xi| \ll R$ respectively, we have that*

$$\|uv\|_{L^2_t L^2_x} = \|u\bar{v}\|_{L^2_t L^2_x} \lesssim R^{-1/2} \|u\|_{0,1/2+} \|v\|_{0,1/2+}.$$

[1] Strictly speaking, we have only shown this for T being an integer multiple of δ, however this can be easily remedied, e.g. by using the fact that the $X^{1,1/2+}$ norm controls the $L^\infty_t H^1_x$ norm on $[T, T+\delta] \times \mathbb{R}$.

Proof. This is an improved Strichartz estimate of the type considered in [3].

It is enough to show that if u and v are solutions of the free Schrödinger equation, that is $u = e^{it\partial_x^2}\phi$ and $v = e^{it\partial_x^2}\psi$, then

$$\|D_x^{1/2}(uv)\|_{L^2} \lesssim \|\phi\|_{L^2}\|\psi\|_{L^2}., \tag{22}$$

where D_x is the operator such that $\widehat{D_x f}(\xi) = \langle\xi\rangle\hat{f}(\xi)$. If we use duality and the change of variable $\xi_1 + \xi_2 = s$ and $|\xi_1|^2 + |\xi_2|^2 = r$, the left hand side of (22) becomes

$$\begin{aligned} &\sup_{\|F\|_{L^2}\le 1} \int R^{1/2} F(\xi_1+\xi_2, |\xi_1|^2+|\xi_2|^2)\hat{\phi}(\xi_1)\hat{\psi}(\xi_2)d\xi_1 d\xi_2 \\ \lesssim &\int R^{1/2}F(s,r)\frac{H(s,r)}{R}dsdr, \end{aligned}$$

where $H(s,r)$ denotes the product of $\hat{\phi}$ and $\hat{\psi}$ in the new variables. Notice that the change of variables introduced above has a Jacobian of size R. Now if we use Cauchy-Schwarz and we change the variables back to ξ_1 and ξ_2, we obtain (22). □

In one of our sub-cases, we shall also take advantage of a trick (originally due to Bourgain [2]) of splitting the symbol $|\xi_1|^2 - \ldots + |\xi_n|^2$ as a sum of $\tau_j \mp |\xi_j|^2$.

Our estimates are not best possible, and it is likely that one can improve the N^{-1+} gain in our estimates, probably to $N^{-3/2+}$. However this will fall short of the N^{-2+} gain needed to push the global well-posedness down to match the local well-posedness theory at $s > 1/2$. However one can recover this by adding higher order correction terms to the energy $E_N(w(t))$, as in [8]. If one does this, one will end up estimating Λ_6 and Λ_8 expressions rather than Λ_4. This will be beneficial because such expressions will have fewer derivatives in their symbol and can therefore enjoy better decay in N. The details of this argument will appear in a later paper.

We set out some notation. Let $n = 4$, 6, or 8, and let $\xi_1, \ldots, \xi_n$ be frequencies such that $\xi_1 + \ldots + \xi_n = 0$. Define $N_i := |\xi_i|$, and $N_{ij} := |\xi_{ij}|$. We adopt the notation that

$$1 \le soprano, alto, tenor, baritone \le n$$

are the distinct indices such that

$$N_{soprano} \ge N_{alto} \ge N_{tenor} \ge N_{baritone}$$

are the highest, second highest, third highest, and fourth highest values of the frequencies $N_1, \ldots, N_n$ respectively (if there is a tie in frequencies, we break the tie arbitrarily).

Since $\xi_1 + \ldots + \xi_n = 0$, we must have $N_{soprano} \sim N_{alto}$. Also, from Proposition 4.1 we see that M_n vanishes unless $N_{soprano} \gtrsim N$.

8. Proof of Lemma 6.1 when $n = 4$

We now estimate the Λ_4 expression. We begin by estimating the multiplier M_4.

Lemma 8.1. *Let $\xi_1, \xi_2, \xi_3, \xi_4$ be such that $\xi_{1234} = 0$.*

- *If $N_{tenor} \sim N_{soprano}$, then*

$$|M_4(\xi_1,\xi_2,\xi_3,\xi_4)| \lesssim N^{-1}(N/N_{soprano})^{1/10}\langle\xi_{12}\xi_{14}\rangle^{1/2}\prod_{j=1}^{4}\langle\xi_j\rangle m(\xi_j). \tag{23}$$

- *If $N_{tenor} \ll N_{soprano}$, then*

$$|M_4(\xi_1,\xi_2,\xi_3,\xi_4)| \lesssim N^{-1}(N/N_{soprano})^{1/10}N_{soprano}\prod_{j=1}^{4}\langle\xi_j\rangle m(\xi_j). \tag{24}$$

Proof. Fix $\xi_1, \dots, \xi_4$. If $N_{soprano} \ll N$ then M_4 vanishes by the second part of Proposition 4.1, so we will assume that $N_{soprano} \gtrsim N$.

We split $M_4 = C_1 M_4' + C_2 M_4''$, where

$$M_4' := m_1 m_2 m_3 m_4 \xi_{12} \xi_{13} \xi_{14}$$

and

$$M_4'' := m_1^2 \xi_1^2 \xi_3 + m_2^2 \xi_2^2 \xi_4 + m_3^2 \xi_3^2 \xi_1 + m_4^2 \xi_4^2 \xi_2.$$

In the $N_{soprano} \gtrsim N$ case we will not need to exploit cancellation between M_4' and M_4'' (although such cancellation certainly exists), and shall estimate them separately.

Let us first prove (23). We begin with estimating M_4'. We have

$$\begin{aligned}
|M_4'| &= N_{12} N_{13} N_{14} m(N_1) m(N_2) m(N_3) m(N_4) \\
&\lesssim N_{12} N_{14} N_{soprano} m(N_{soprano})^3 \\
&\lesssim \langle N_{12} N_{14} \rangle^{1/2} N_{soprano}^2 m(N_{soprano})^3 \\
&\lesssim \langle N_{12} N_{14} \rangle^{1/2} \frac{1}{N} (N/N_{soprano})^{1/10} \langle N_{soprano} \rangle^3 m(N_{soprano})^3 \\
&\sim N^{-1} (N/N_{soprano})^{1/10} \langle N_{12} N_{14} \rangle^{1/2} \langle N_{soprano} \rangle m(N_{soprano}) \langle N_{alto} \rangle m(N_{alto}) \langle N_{tenor} \rangle m(N_{tenor}) \\
&\lesssim N^{-1} (N/N_{soprano})^{1/10} \langle N_{12} N_{14} \rangle^{1/2} \prod_{j=1}^4 \langle N_j \rangle m(N_j)
\end{aligned}$$

as desired.

It remains to estimate M_4''. We divide into two cases: $N_{baritone} \sim N_{soprano}$ and $N_{baritone} \ll N_{soprano}$.

Case 1: $N_{baritone} \sim N_{soprano}$.

In this case all the frequencies are comparable to each other. By symmetry we may assume that $N_{12} \leq N_{14}$, in which case it suffices to show

$$|M_4''| \lesssim N^{-1} (N/N_1)^{1/10} N_{12} N_1^4 m(N_1)^4.$$

We can rewrite $M_4'' = f(0) - f(h)$, where

$$f(h) := m(\xi_1 - h)^2 (\xi_1 - h)^2 (\xi_3 + h) + m(\xi_3 + h)^2 (\xi_3 + h)^2 (\xi_1 - h)$$

and $h := \xi_1 + \xi_2$. A routine calculation shows that

$$|f'(x)| \lesssim m(N_1)^2 N_1^2$$

for all $x = O(N_1)$, so by the mean value theorem and the assumption $N_1 \gtrsim N$ we have

$$|M_4''| = |f(0) - f(h)| \lesssim N_{12} m(N_1)^2 N_1^2 \lesssim N^{-1} (N/N_1)^{1/10} N_{12} N_1^4 m(N_1)^4$$

as desired (in fact we gain an additional power of N).

Case 2: $N_{baritone} \ll N_{soprano}$.

By symmetry we may assume that *baritone* $= 4$, thus $N_1 \sim N_2 \sim N_3 \gg N_4$. In this case $N_{14} \sim N_1$, $N_{12} = N_{34} \sim N_1$ and $\langle N_4 \rangle m(N_4) \gtrsim 1$, so it suffices to show

$$|M_4''| \lesssim N^{-1} (N/N_1)^{1/10} N_1^4 m(N_1)^3.$$

But we may crudely estimate the left-hand side by

$$|M_4''| \lesssim m(N_1)^2 N_1^3 + m(N_4)^2 N_4^2 N_1 \lesssim m(N_1)^2 N_1^3$$

which suffices since $N_1 \gtrsim N$. This proves (23).

Now we show (24). Observe that

$$N_{12} N_{13} N_{14} \lesssim N_{soprano}^2 N_{tenor}$$

and hence

$$\begin{aligned}|M_4'| &\lesssim N_{soprano}^2 N_{tenor} m(N_{soprano}) m(N_{alto}) m(N_{tenor}) m(N_{baritone}) \\ &\lesssim \prod_{j=1}^4 \langle N_j \rangle m(N_j) \\ &\lesssim N^{-1+} N_{soprano}^{1-} \prod_{j=1}^4 \langle N_j \rangle m(N_j).\end{aligned}$$

Thus it only remains to estimate M_4''. Since $\langle N_{baritone} \rangle m(N_{baritone})$ and $m(N_{tenor}) N^{-1} (N/N_{soprano})^{1/10} N_{soprano}$ are both $\gtrsim 1$, it suffices to show

$$|M_4''| \lesssim m(N_{soprano})^2 N_{soprano}^2 N_{tenor}.$$

By symmetry we may reduce to one of two cases.

Case 1: $N_3 = N_{tenor}$ **and** $N_4 = N_{baritone}$.

We crudely estimate

$$|M_4''(\xi_1, \xi_2, \xi_3, \xi_4)| \lesssim m(N_1)^2 N_1^2 N_3 + m(N_2)^2 N_2^2 N_4 + m(N_3)^2 N_3^2 N_1 + m(N_4)^2 N_4^2 N_2 \lesssim m(N_1)^2 N_1^2 N_3$$

as desired.

Case 2: $N_2 = N_{tenor}$ **and** $N_4 = N_{baritone}$.

In this case we estimate

$$\begin{aligned}|M_4''| &\lesssim |m_1^2 \xi_1^2 \xi_3 + m_3^2 \xi_3^2 \xi_1| + m(N_2)^2 N_2^2 N_4 + m(N_4)^2 N_4^2 N_2 \\ &= N_1 N_3 |m(\xi_1)^2 \xi_1 - m(\xi_1 + \xi_2 + \xi_4)^2 (\xi_1 + \xi_2 + \xi_4)| + O(m(N_1)^2 N_1^2 N_2).\end{aligned}$$

The function $m(\xi_1 + h)^2 (\xi_1 + h)$ has a derivative of $O(m(N_1)^2)$ whenever $|h| \ll N_1$, thus by the mean value theorem we thus have

$$|M_4'(\xi_1, \xi_2, \xi_3, \xi_4)| \lesssim N_1 N_3 N_{24} m(N_1)^2 + O(m(N_1)^2 N_1^2 N_2) \lesssim m(N_1)^2 N_1^2 N_2$$

as desired. □

We now prove (20) in the $n = 4$ case. It suffices to show that

$$\int_T^{T+\delta} \Lambda_4(M_4; w_1(t), \overline{w_2(t)}, w_3(t), \overline{w_4(t)})\, dt \lesssim N^{-1+} \prod_{j=1}^4 \|I w_j\|_{1,1/2+}$$

for all Schwartz functions $w_1, \ldots, w_4$ on $\mathbb{R} \times \mathbb{R}$. Since M_4 vanishes for $N_{soprano} \ll N$, it suffices by dyadic decomposition to show that

$$\int_T^{T+\delta} \Lambda_4(M_4 \chi_{N_{soprano} \sim 2^k}; w_1(t), \overline{w_2(t)}, w_3(t), \overline{w_4(t)})\, dt \lesssim N^{-1+} 2^{(0+)k} (N/2^k)^{1/10} \prod_{j=1}^4 \|I w_j\|_{1,1/2+}$$

for all integers k for which $2^k \gtrsim N$. (The exact choice of the cutoff $\chi_{N_{soprano} \sim 2^k}$ is not important as we shall soon be taking absolute values everywhere anyway).

Fix k. Without loss of generality we may assume that the Fourier transforms $\tilde{w}_j$ are real and non-negative. We divide into the Λ_4 integral into the regions $N_{tenor} \sim N_{soprano}$ and $N_{tenor} \ll N_{soprano}$.

Case 1. $N_{tenor} \sim N_{soprano}$.

We first perform some manipulations to eliminate the cutoff $\chi_{[T,T+\delta]}(t)$. Write $\chi_{[T,T+\delta]}(t) = a(t) + b(t)$, where $a(t)$ is $\chi_{[T,T+\delta]}(t)$ convolved with a smooth approximation to the identity of width 2^{-100k}, and $b(t) = \chi_{[T,T+\delta]}(t) - a(t)$.

Let us first consider the contribution of $b(t)$. We crudely estimate $M_4 = O(2^{10k})$ and estimate this contribution by

$$2^{10k} \int\int |b(t)||w_1(t,x)||w_2(t,x)||w_3(t,x)||w_4(t,x)|\ dxdt.$$

By Hölder, three applications of (3), one application of (4), and four applications of (15) we can bound this by

$$2^{10k}\|b\|_2 \prod_{j=1}^4 \|Iw_j\|_{1,1/2+}.$$

Since $\|b\|_2 \lesssim 2^{-50k}$, the claim then follows.

Now consider the contribution of $a(t)$. We use

Lemma 8.2. *We have*

$$\|a(t)w_1\|_{1,1/2+} \lesssim 2^{(0+)k}\|w_1\|_{1,1/2+}.$$

Proof. By applying Plancherel, restricting to a single frequency ξ, and then undoing Plancherel, we see that it suffices to show that

$$\|a(t)f\|_{H_t^{1/2+}} \lesssim 2^{(0+)k}\|f\|_{H_t^{1/2+}}$$

for all functions f. But this follows from the routine calculation

$$\|a(t)\|_{H_t^{1/2+}} \lesssim 2^{(0+)k}$$

and the fact that $H_t^{1/2+}$ is closed under multiplication. □

It therefore suffices to show

$$|\int \Lambda_4(M_4\chi_{N_{tenor}\sim N_{soprano}\sim 2^k}; w_1(t), \overline{w_2(t)}, w_3(t), \overline{w_4(t)})\ dt| \lesssim N^{-1}(N/2^k)^{1/10}\prod_{j=1}^4 \|Iw_j\|_{1,1/2+}.$$

Without loss of generality we may assume that the Fourier transforms $\tilde{w}_j$ are real and non-negative. By Plancherel and (23) we estimate the left-hand side by

$$N^{-1}(N/2^k)^{1/10}|\int_* \langle\xi_{12}\xi_{14}\rangle^{1/2}\widetilde{ID_xw_1}(\tau_1,\xi_1)\widetilde{\overline{ID_xw_2}}(\tau_2,\xi_2)\widetilde{ID_xw_3}(\tau_3,\xi_3)\widetilde{\overline{ID_xw_4}}(\tau_4,\xi_4)|.$$

From the identity (cf. Bourgain [2] and Kenig-Ponce-Vega [17])

$$\begin{aligned}\sum_{j=1}^4(\tau_j - (-1)^{j-1}\xi_j^2) &= -\xi_1^2 + \xi_2^2 - \xi_3^2 + \xi_4^2 \\ &= \xi_{12}\xi_{2-1} + \xi_{34}\xi_{4-3} \\ &= \xi_{12}(\xi_{2-1} - \xi_{4-3}) \\ &= -2\xi_{12}\xi_{14}\end{aligned}$$

we see that

$$\langle\xi_{12}\xi_{14}\rangle \lesssim \langle\tau_j - (-1)^{j-1}\xi_j^2\rangle$$

for some $j = 1,2,3,4$. We shall assume $j = 1$; the argument for other values of j is similar. We can then use duality and Plancherel to estimate the previous by

$$N^{-1}(N/2^k)^{1/10}\|Iw_1\|_{1,1/2+}\|\overline{ID_xw_2}ID_xw_3\overline{ID_xw_4}\|_{L_t^2L_x^2}.$$

But this is acceptable by Hölder and three applications of (3).

Case 2. $N_{tenor} \ll N_{soprano}$.

We shall assume that $soprano = 1$ and $alto = 2$; the reader may verify that the other cases follow by the same argument. We may then restrict w_1, w_2 to have Fourier support in $|\xi| \sim 2^k$ and w_3, w_4 to have Fourier support in the region $|\xi| \ll 2^k$.

By (24) we have

$$|M_4| \lesssim N^{-1}(N/2^k)^{1/10} 2^k \prod_{j=1}^{4} \langle \xi_i \rangle m(\xi_i).$$

The claim then follows from Hölder and two applications of Proposition 7.1.

9. Proof of Lemma 6.1 when $n = 6$

We begin with the analogue of Lemma 8.1.

Lemma 9.1. *Let $\xi_1, \dots, \xi_6$ be such that $\xi_{123456} = 0$.*

- *If $N_{tenor} \sim N_{soprano}$, then*

$$|M_6(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)| \lesssim N^{-1} \langle \xi_{soprano} \rangle m(\xi_{soprano}) \langle \xi_{alto} \rangle m(\xi_{alto}) \langle \xi_{tenor} \rangle m(\xi_{tenor}). \tag{25}$$

- *If $N_{tenor} \ll N_{soprano}$, then*

$$|M_6(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)| \lesssim N^{-1+} \langle N_{soprano} \rangle^{1-} \langle \xi_{soprano} \rangle m(\xi_{soprano}) \langle \xi_{alto} \rangle m(\xi_{alto}). \tag{26}$$

One can improve these estimates by exploiting further cancellation in the expression M_6, but we shall not need to do so because of the good smoothing properties of our equation (6).

Proof. Since $\xi_{123456} = 0$, we have $N_{alto} \sim N_{soprano}$. We may also assume that $N_{soprano} \gtrsim N$ since M_6 vanishes otherwise.

We have the very crude estimate

$$|M_6| \lesssim N_{soprano}^2.$$

If $N_{tenor} \sim N_{soprano}$, we then have

$$|M_6| \lesssim N_{soprano}^2 \lesssim N^{-1} m(N_{soprano}) N_{soprano} m(N_{alto}) N_{alto} m(N_{tenor}) N_{tenor}$$

(using the hypothesis $s > 2/3$), and (25) follows.

Now suppose that $N_{tenor} \ll N_{soprano}$. Then

$$|M_6| \lesssim N_{soprano}^2 \lesssim N^{-1+} \langle N_{soprano} \rangle^{1-} m(N_{soprano}) N_{soprano} m(N_{alto}) N_{alto}$$

(since $s > 1/2$), and (26) follows. □

We now prove (20) for $n = 6$. As in the previous section it suffices to show

$$\int_T^{T+\delta} \Lambda_6(M_6; w_1(t), \overline{w_2(t)}, w_3(t), \overline{w_4(t)}, w_5(t), \overline{w_6(t)})\ dxdt \lesssim N^{-1+} \prod_{j=1}^{6} \|Iw_j\|_{1,1/2-}$$

for all Schwartz functions $w_1, \dots, w_6$ on $\mathbb{R} \times \mathbb{R}$. Without loss of generality we may assume that the Fourier transforms $\tilde{w}_i$ of w_i are real and non-negative.

We again divide into the cases $N_{tenor} \sim N_{soprano}$ and $N_{tenor} \ll N_{soprano}$.

Case 1. $N_{tenor} \sim N_{soprano}$.

By (25) and symmetry it suffices to show

$$\int_T^{T+\delta} \int \prod_{j=1}^{3} |D_x I w_j| \prod_{j=4}^{6} |w_j|\ dxdt \lesssim \prod_{j=1}^{6} \|Iw_j\|_{1,1/2+}.$$

But this follows from Hölder, six applications of (3) first, and three applications of (15) after.

Case 2. $N_{tenor} \ll N_{soprano}$.

We shall assume that *soprano* $= 1$ and *alto* $= 2$; the reader may verify that the other cases follow by the same argument.

First suppose that $N_{soprano} \sim 2^k$ for some integer k. Then w_1, w_2 have Fourier support on $|\xi| \sim 2^k$, while w_3, w_4, w_5, w_6 have Fourier support on $|\xi| \ll 2^k$.

We apply (26), and bound the contribution of this case by

$$N^{-1+}2^{(1-)k} \int_T^{T+\delta} \int \prod_{j=1}^{2} |D_x I w_j| \prod_{j=3}^{6} |w_j| \; dxdt,$$

which we bound using Hölder by

$$N^{-1+}2^{(1-)k} \|(D_x I w_1) w_3\|_{L_t^2 L_x^2} \|(D_x I w_2) w_4\|_{L_t^2 L_x^2} \|w_5\|_{L_t^\infty L_x^\infty} \|w_6\|_{L_t^\infty L_x^\infty}.$$

By Lemma 7.1, (5), and (15) we can bound this by

$$N^{-1+}2^{(0-)k} \prod_{j=1}^{6} \|I w_j\|_{1,1/2+}$$

The claim then follows by summing in k.

10. Proof of Lemma 6.1 when $n = 8$

We begin with the analogue of Lemma (8.1).

Lemma 10.1. *For any $\xi_1, \ldots, \xi_6$ with $\xi_{123456} = 0$, we have*

$$|M_8(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7, \xi_8)| \lesssim N^{-1} \langle \xi_{soprano} \rangle m(\xi_{soprano}) \langle \xi_{alto} \rangle m(\xi_{alto}). \tag{27}$$

Proof. As usual we may assume that $N_{soprano} \sim N_{alto} \gtrsim N$. We crudely estimate

$$|M_8| \lesssim N_{soprano} \lesssim N^{-1} N_{soprano} m(N_{soprano}) N_{alto} m(N_{alto})$$

and the claim follows. □

To prove (20) for $n = 8$ it suffices to show

$$\int_T^{T+\delta} \Lambda_8(M_8; w_1(t), \ldots, \overline{w_8(t)}) \; dt \lesssim N^{-1} \prod_{j=1}^{8} \|I w_j\|_{1,1/2+}$$

for all Schwartz functions $w_1, \ldots, w_8$ on $\mathbb{R} \times \mathbb{R}$. Without loss of generality we may assume that the Fourier transforms $\tilde{w}_i$ of w_i are real and non-negative. By Lemma 10.1 and symmetry it thus suffices to show

$$\int_T^{T+\delta} \int |D_x I w_1| |D_x I w_2| \prod_{j=3}^{8} |w_j| \; dxdt \lesssim N^{-1} \prod_{j=1}^{8} \|I w_j\|_{1,1/2+}.$$

But this follows from Hölder, six applications of (3), and two applications of (5) and (15).

Remark 10.2. As it was shown in Section 3, the gauge transform in Definition 3.1 introduces a quintic term in the initial value problem (6). Then one can ask if the same arguments we proposed above can be used in order to study the global well-posedness of the quintic nonlinear Schrödinger initial value problem

$$\begin{cases} i\partial_t v + \partial_x^2 v + \lambda |v|^4 v = 0, \\ v(x,0) = v_0(x), \qquad x \in \mathbb{R}, t \in \mathbb{R}, \end{cases} \tag{28}$$

where $\lambda \in \mathbb{R}$. In this case we define the energy

$$H(f) := \int |\partial_x f(x)|^2 \, dx - \frac{\lambda}{6} \int |f|^2 \, dx.$$

By Plancherel, we may write $H(f)$ using the Λ notation as

$$H(f) = \Lambda_2(\xi_1 \xi_2; f) - \frac{\lambda}{6} \Lambda_6(1; f).$$

As in Lemma 3.5 one can prove that the energy $H(v(t))$ of the solution v for (28) is constant. Now let's define the new energy

$$H_N(v) = H(Iv) = \Lambda_2(\xi_1 \xi_2 m_1 m_2; v) - \frac{\lambda}{6} \Lambda_6(\prod_{i=1}^{6} m_i; v).$$

just like we did in Secition 4. Then by the analogue of (8) $\partial_t H_N(v(t))$ will involve terms of type Λ_2, Λ_6 and Λ_{10}. Using the same ideas presented in the proof of Lemma 6.1 we can estimate in the appropriate way also the term involving Λ_{10}. If in (28) we assume that $\lambda < 0$ (defocusing) or that the L^2 norm of the initial data is small (so that the Gagliardo-Niremberg inequality can be applied) then the energy $H(v)(t)$ stays positive for all times and global well-posedness in H^s for $s > 2/3$ will follow. We will present the details of the proof in a future paper. It has to be said here that global results for "small data" are already available for (28) through more standard arguments [6].

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