Math 5587 Midterm II Information

- The midterm will take place on Thursday, November 3, during class.
- The exam will cover everything up to and including the lecture on Thurs October 20.
- The exam is closed book. No textbooks, notes, or calculators are allowed. The formula sheet below will be provided on the exam.
- The exam will have 5 questions. The first 3 will be short, and the last 2 will be longer and slightly more involved. Below are a collection of sample midterm questions for you to practice.

Sample questions

1. Determine whether the following statements are true or false. No justification is required.

   (a) The Fourier series for any function $f : [-\pi, \pi] \to \mathbb{R}$ converges uniformly.

   (b) The Fourier series for a function f converges uniformly provided $\int_{-\pi}^{\pi} f(x)^2\, dx < \infty$.

   (c) The Fourier series for a discontinuous function does not converge pointwise due to Gibb's phenomenon.

   (d) Plancheral's identity is

   $$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)^2\, dx = \sum_{n=-\infty}^{\infty} |c_n|^2 \quad \text{where } c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}.$$

   (e) If a sequence of functions f_n converges in norm to f, then f_n converges uniformly to f.

   (f) A Fourier series always has an infinite number of nonzero terms.

2. Let $f(x) = \exp(\sin(x))\sin(x)$. Explain how you can deduce, without any computations, that $A_n = 1/n^2$ and $B_n = 1/n^4$ cannot be the coefficients of the Fourier series for f

   $$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(nx) + B_n \sin(nx).$$

3. Show that

   $$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

   [Hint: Find the complex version of the Fourier series for $f(x) = x$ on $[-\pi, \pi]$ and compute both sides of Plancheral's identity

   $$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)^2\, dx = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

   ]

4. Use the same idea from 2 to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

[Hint: Use $f(x) = x^2$.]

5. Solve the heat equation $u_t = u_{xx}$ on the rectangle $0 < x < \pi$, $t > 0$, with homogeneous Neumann boundary conditions $u_x(0,t) = u_x(\pi,t) = 0$ and initial condition $u(x,0) = x$.

6. Solve the heat equation $u_t = u_{xx}$ on the rectangle $0 < x < \pi$, $t > 0$, with mixed boundary conditions $u_x(0,t) = u(\pi,t) = 0$ and initial condition $u(x,0) = x$.

7. Solve the heat equation $u_t - u_{xx} = 0$ on the rectangle $0 < x < \pi$ and $t > 0$ with initial condition $u(x,0) = 1$ and Dirichlet boundary conditions $u(0,t) = u(\pi,t) = 0$ for $t > 0$.

8. Solve the wave equation $u_{tt} = u_{xx}$ on the rectangle $-\pi < x < \pi$, $t > 0$ with periodic boundary conditions $u(-\pi,t) = u(\pi,t)$ and $u_x(-\pi,t) = u_x(\pi,t)$, initial position $u(x,0) = x$ and initial velocity $u_t(x,0) = 0$.

9. Solve the heat equation $u_t - u_{xx} = 0$ on the rectangle $0 < x < \pi$ and $0 < t < \infty$ with homogeneous Neumann boundary conditions $u_x(0,t) = u_x(\pi,t) = 0$ for $t > 0$ and initial condition $u(x,0) = x + \cos(2x)$.

10. Solve the wave equation $u_{tt} = u_{xx}$ on the rectangle $0 < x < \pi$, $t > 0$ with mixed boundary conditions $u(0,t) = u_x(\pi,t) = 0$, initial position $u(x,0) = 0$ and initial velocity $u_t(x,0) = 1$.

11. Define the sequence of functions $f_n(x) = x^n$ for $0 \le x \le 1$. Show that f_n converges pointwise on $[0,1]$ to f defined by

$$f(x) = \begin{cases} 0, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x = 1. \end{cases}$$

Does f_n converge to f uniformly on $[0,1]$? Does f_n converge in norm? Justify your answer.

12. Suppose that $f : [a,b] \to \mathbb{R}$ is continuously differentiable.

(a) For $a \le x < y \le b$ show that

$$\int_x^y |f'(t)|\, dt \le \|f'\| |x - y|^{\frac{1}{2}}.$$

[Hint: Use the Cauchy-Schwarz inequality $(f', g) \le \|f'\|\|g\|$ with $g = 1$.]

(b) Use part (a) to show that

$$|f(x) - f(y)| \le \|f'\| |x - y|^{\frac{1}{2}},$$

for all $x, y \in [a,b]$.

13. Suppose that $f : [a,b] \to \mathbb{R}$ is twice continuously differentiable and $f(a) = f(b) = 0$. Prove the interpolation inequality

$$\|f'\|^2 \leq \|f\|\|f''\|.$$

[Hint: Write out the integral on the left, and integrate by parts. Then apply Cauchy-Schwarz $(f,g) \leq \|f\|\|g\|$.]

14. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is 2π-periodic and k-times continuously differentiable. Show that

$$f^{(k)}(x) = \sum_{n=-\infty}^{\infty} c_n (in)^k e^{inx},$$

where $f^{(k)}$ denotes the k^{th} derivative of f, and c_n are the Fourier coefficients of f, given by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx.$$

Formula Sheet

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nx) + B_n \sin(nx) = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\, dx = \int_{-\pi}^{\pi} \sin(nx)\sin(mx)\, dx = \begin{cases} \pi, & \text{if } n = m \\ 0, & \text{otherwise.} \end{cases}.$$

$$\int_{-\pi}^{\pi} \cos(nx)\sin(mx)\, dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} e^{inx}e^{-imx}\, dx = \begin{cases} 2\pi, & \text{if } n = m \\ 0, & \text{otherwise.} \end{cases}.$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)^2\, dx = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\, dx = \frac{1}{2}A_0^2 + \sum_{n=1}^{\infty} A_n^2 + B_n^2.$$