Math 5587 – Lecture 3

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1 Energy methods for the wave equation

Consider a finite string of length $l > 0$, density $\rho > 0$, and tension $\kappa > 0$. As usual, let $u(x,t)$ denote the displacement from equilibrium, and suppose we fix the ends of the string at $x = 0$ and $x = l$. This would model a piano or guitar string, for instance. This corresponds to homogeneous Dirichlet boundary conditions $u(0,t) = u(l,t) = 0$ for all $t \geq 0$. Then the displacement $u(x,t)$ satisfies the wave equation

$$\begin{cases} \rho u_{tt} - \kappa u_{xx} = 0, & \text{for } 0 < x < l \\ u(0,t) = u(l,t) = 0, & \text{for } t \geq 0. \end{cases} \tag{1}$$

From physics, we know that the total energy in the string is made up of two terms: kinetic energy and potential energy. The total kinetic energy of the string at time t is

$$\mathrm{KE}(t) = \frac{1}{2}\rho \int_0^l u_t(x,t)^2 \, dx,$$

while the total potential energy of the string is

$$\mathrm{PE}(t) = \frac{1}{2}\kappa \int_0^l u_x(x,t)^2 \, dx.$$

The total energy of the string is

$$\mathrm{E}(t) = \mathrm{KE}(t) + \mathrm{PE}(t) = \frac{1}{2}\int_0^l (\rho u_t^2 + \kappa u_x^2) \, dx.$$

While the kinetic and potential energy may vary over time, we should expect by conservation of energy that the total energy should be constant in time.

To check this, we differentiate $\mathrm{E}(t)$ in t:

$$\begin{aligned} \frac{d\mathrm{E}}{dt} &= \int_0^l (\rho u_t u_{tt} + \kappa u_x u_{xt}) \, dx \\ &= \int_0^l \rho u_t u_{tt} \, dx + \int_0^l \kappa u_x u_{xt} \, dx \\ &= \int_0^l \rho u_t u_{tt} \, dx + \kappa u_x u_t \Big|_0^l - \int_0^l \kappa u_{xx} u_t \, dx \\ &= \int_0^l u_t(\rho u_{tt} - \kappa u_{xx}) \, dx + \kappa u_x(l,t) u_t(l,t) - \kappa u_x(0,t) u_t(0,t) \\ &= 0. \end{aligned}$$

Notice we performed integration by parts in the third line. The first term in the last line is zero because u satisfies the wave equation (1). The Dirichlet boundary conditions $u(0,t) = u(l,t) = 0$ imply that $u_t(0,t) = u_t(l,t) = 0$, which makes the second term in the final line vanish. Since $d\mathrm{E}/dt = 0$, the total energy $\mathrm{E}(t)$ is constant in time, as expected.

Exercise 1. Suppose you did not know the exact formulas for kinetic and potential energy, but that you suspect that

$$\mathrm{KE}(t) = a\int_0^l u_t(x,t)^2\,dx \quad \text{and} \quad \mathrm{PE}(t) = b\int_0^l u_x(x,t)^2\,dx,$$

for some constants a and b. Show that

$$\frac{d\mathrm{E}}{dt} = 0 \text{ if and only if } \frac{b}{a} = \frac{\kappa}{\rho}.$$

Thus, the wave equation can help us deduce formulas for physical quantities, like energy. A common technique in PDE theory is to look for a quantity, like energy, that is conserved or decreased by the PDE.

Exercise 2. Suppose the string is subject to homogeneous Neumann boundary conditions, thus u satisfies

$$\begin{cases} \rho u_{tt} - \kappa u_{xx} = 0, & \text{for } 0 < x < l \\ u_x(0,t) = u_x(l,t) = 0, & \text{for } t \geq 0. \end{cases} \tag{2}$$

Recall this corresponds to allowing the ends of the string to move freely along a frictionless vertical track. Show that the total energy

$$\mathrm{E}(t) = \frac{1}{2}\int_0^l (\rho u_t^2 + \kappa u_x^2)\,dx$$

is conserved.

2 Uniqueness and stability via energy methods

Energy methods are very powerful tools for showing uniqueness and stability of solutions of PDE. We illustrate energy methods for the wave equation below, but we will see that similar ideas hold for the heat equation shortly.

Let $u^1(x,t)$ and $u^2(x,t)$ be two solutions of the wave equation (1). For simplicity, let us take $\rho = \kappa = 1$. Since the wave equation is linear, $w(x,t) := u^1(x,t) - u^2(x,t)$ is also a solution of the same wave equation (1). Since the energy for w is constant in time, we have

$$\int_0^l w_t(x,t)^2 + w_x(x,t)^2\,dx = \int_0^l w_t(x,0)^2 + w_x(x,0)^2\,dx \tag{3}$$

for all t. Suppose we have the initial conditions

$$u^i(x,0) = f_i(x),\ u_t^i(x,0) = g_i(x) \quad \text{for} \quad i = 1,2.$$

Integrating both sides of (3) from $t = 0$ to $t = T$ and substituting the initial conditions gives

$$A := \int_0^T \int_0^l (u_t^1 - u_t^2)^2 + (u_x^1 - u_x^2)^2 \, dx \, dt = T \int_0^l (g_1 - g_2)^2 + (f_1' - f_2')^2 \, dx =: B. \quad (4)$$

The equation above proves both stability and uniqueness for the wave equation (1). To see this, notice that if the initial conditions for u^1 and u^2 agree at $t = 0$, i.e., $g_1 = g_2$ and $f_1 = f_2$, then $B = 0$ and so $A = 0$ as well. Thus $u_t^1 = u_t^2$ and $u_x^1 = u_x^2$, and so $u^1 = u^2$. This proves that there is at most one solution with fixed initial conditions (i.e. uniqueness).

Equation (4) can also be viewed as a stability result. If g_1 is close to g_2 and f_1 is close to f_2, then B is small, and so A is small as well. The quantity A is a measure of how close u^1 and u^2 are, so when A is small, u^1 and u^2 are close as well (more precisely, their derivatives are close in the square integrable sense). So small changes in the initial conditions yield correspondingly small changes in the solutions.

The final piece of information we need to establish well-posedness of the wave equation is existence of a solution. We will return to this later in the course after studying Fourier series.