Math 5587 – Lecture 2

Jeff Calder

August 31, 2016

1 Initial/boundary conditions and well-posedness

1.1 ODE vs PDE

Recall that the general solutions of ODEs involve a number of arbitrary constants.

Example 1.1. Consider the simple ODE $u'(x) + 2xu(x) = 0$. What is the general solution? Your answer should involve an arbitrary constant C.

In contrast, the general solutions of PDE involve one or more arbitrary *functions*.

Example 1.2. Consider the PDE

$$u_{xy}(x, y) + 2xu_y(x, y) = 0.$$

What is the general solution? Your answer should involve two arbitrary functions $F(x)$ and $G(y)$.

To single out one solution, we need to impose boundary and/or initial conditions. The conditions are usually motivated by physics, and depend on the particular PDE.

1.2 Initial conditions

When the PDE involves time, one often specifies an initial condition of the form

$$u(x, t_0) = f(x),$$

where $f(x)$ is a given function and t_0 is the initial time (usually $t_0 = 0$). For diffusion equations, f represents the concentration or heat distribution at time t_0. For the wave equation, there is a pair of initial conditions

$$u(x, t_0) = f(x) \quad \text{and} \quad u_t(x, t_0) = g(x),$$

where f is the initial position and g is the initial velocity of the string.

1.3 Boundary conditions

Physical problems usually have a domain $D \subseteq \mathbb{R}^n$ in which the PDE is valid. It is physically obvious that we need to specify some type of boundary condition. For instance, a metal rod submerged in a bath of water will diffuse heat differently near the boundary than one that is suspended in air, or insulated in cork. The interpretation of the boundary condition depends on the physical quantities being described by the PDE.

1.3.1 The heat equation: Neumann, Dirichlet, and Robin conditions

Let us first consider the one dimensional heat equation

$$u_t - ku_{xx} = 0 \quad \text{for } 0 < x < l,$$

describing the diffusion of heat within a rod of length l. We always assume the length of the rod is insulated, so that heat can only flow to the left and right. Let's now consider what happens at the ends of the rod.

First suppose the entire rod is insulated, including the ends. Then heat cannot enter or leave the rod and the total heat is conserved. Furthermore, the heat flux at $x = 0$ and $x = l$ must be zero, since heat cannot flow into or out of the ends of the rod. Recall that by Fourier's law of heat conduction

$$\text{Heat Flux} = F(x,t) = -ku_x(x,t),$$

where $k > 0$ is the thermal conductivity. Then $F(0,t) = 0 = F(l,t)$ and hence

$$\boxed{u_x(0,t) = 0 = u_x(l,t) \quad \text{for all } t \geq 0.} \tag{1}$$

The boundary conditions (1) are called *homogeneous Neumann conditions*. For heat diffusion within a 2D plate or a 3D body, the corresponding homogeneous Neumann conditions are

$$\nabla u(x,y,z,t) \cdot \mathbf{n} = 0 \quad \text{for all } t \geq 0 \text{ and } (x,y,z) \in \partial D,$$

where $\mathbf{n}$ is the outward unit normal vector to the domain $D \subseteq \mathbb{R}^n$, and ∂D is the boundary of D. Recall the quantity on the left above is the normal derivative $\partial u/\partial \mathbf{n}$.

Now, suppose we are somehow able to fix the temperature of the rod at $x = 0$ and $x = l$[1]. Then we have the boundary conditions

$$\boxed{u(0,t) = a \text{ and } u(l,t) = b \quad \text{for all } t \geq 0,} \tag{2}$$

which are called *Dirichlet conditions*. The values of a and b are parameters corresponding to the temperature we choose for each end of the rod. It is common to take $a = b = 0$, in which case we call (2) *homogeneous Dirichlet conditions*. In higher dimensions, Dirichlet conditions correspond to specifying $u(x,y,z,t)$ at all points $(x,y,z) \in \partial D$.

To fix the temperature at either end of the rod, we might try submersing one end of the rod in a large reservoir of, say, water, with a fixed temperature b. The submersed end of the rod will not immediately change temperature to reflect the common temperature of the reservoir; instead, Newton's law of cooling states that the rate at which the temperature changes (or the flux) is proportional to the difference in temperature between the end of the rod and the reservoir. We assume the reservoir is so large that its temperature cannot be changed by heat flowing from the rod to the reservoir, and vice versa. Therefore at $x = l$ we have

$$\text{Heat Flux} = -ku_x(l,t) = \beta(u(l,t) - b),$$

where $\beta > 0$. It follows that

$$\boxed{u(l,t) + Bu_x(l,t) = b \quad \text{for all } t \geq 0,} \tag{3}$$

[1] It is not possible to exactly fix the temperature, so you should think of the Dirichlet conditions as an idealization of the Robin condition to follow.

for some $B > 0$ (here $B = k/\beta$). The boundary condition (3) is called the *Robin condition*. When $\beta = \infty$, we have instantaneous heat transfer from the rod to the reservoir, and we recover the Dirichlet condition $u(l,t) = b$ since $B = 0$. When $\beta = 0$ we have no heat transfer between the rod and reservoir, and we recover the Neumann condition $u_x(l,t) = 0$.

At the other end $x = 0$, we have

$$\text{Heat Flux} = -ku_x(0,t) = -\alpha(u(0,t) - a),$$

for $\alpha > 0$. The reason for the minus sign is that heat flowing into the reservoir is now flowing to the left, which is the negative direction. We find the Robin condition here to be

$$\boxed{u(0,t) - Au_x(0,t) = a \quad \text{for all } t \geq 0,} \tag{4}$$

where $A = k/\alpha > 0$.

In higher dimensions, the Robin conditions corresponds to

$$u(x,y,z,t) + A\nabla u(x,y,z,t) \cdot \mathbf{n} = 0 \quad \text{for all } t \geq 0 \text{ and } (x,y,z) \in \partial D.$$

Notice the presence of the outward normal $\mathbf{n}$ accounts for the difference in sign between (3) and (4) (at $x = 0$, $\mathbf{n} = -1$, and at $x = l$, $\mathbf{n} = 1$).

Finally, suppose the rod is bent into a circular ring. Then $x = x_0$ and $x = x_0 + l$ correspond to the same point on the ring. It follows that

$$\boxed{u(0,t) = u(l,t) \ \text{ and } \ u_x(0,t) = u_x(l,t) \quad \text{for all } t \geq 0.} \tag{5}$$

These are called *periodic boundary conditions*.

Remark 1. Notice in each case above, we could have allowed a and b to vary with time $t > 0$, i.e., $a = a(t)$ and $b = b(t)$.

Remark 2. We can also consider mixed boundary conditions, where the condition at $x = 0$ is different than that at $x = l$. For example, if the $x = 0$ end is insulated, and the $x = l$ end is held at zero temperature, then we have the mixed boundary conditions

$$u_x(0,t) = 0 \ \text{ and } \ u(l,t) = 0.$$

1.3.2 The wave equation: Dirichlet, Neumann, and Robin conditions

We now give interpretations of the Dirichlet, Neumann, and Robin conditions for the wave equation

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{for } 0 < x < l.$$

The Dirichlet condition corresponds to fixing the ends of the string at some arbitrary positions

$$u(0,t) = a \ \text{ and } u(l,t) = b.$$

If we allow $a = a(t)$ and $b = b(t)$ to vary with time, then this corresponds to moving the ends of the string in a predefined way. As for the heat equation, we commonly take $a = b = 0$.

Now suppose the ends of the string are attached to a frictionless vertical track. That is, the ends are allowed to move in the vertical direction with no resistance. By Newton's law

$F = ma$, any object with zero mass and finite acceleration must have zero net force acting on it. Recall the vertical component of tension for small vibrations is approximately κu_x where $\kappa > 0$ is the magnitude of the tension vector. Since the end points $x = 0$ and $x = l$ have zero mass, we have

$$\kappa u_x(0, t) = \text{Vertical component of tension} = 0,$$

and $\kappa u_x(l, t) = 0$. We have recovered the Neumann conditions

$$u_x(0, t) = 0 = u_x(l, t).$$

If this argument seems a bit sketchy to you, here is a more rigorous version. Instead of considering the point $x = 0$, let us consider a small segment of string between $x = 0$ and $x = h$, for some $h > 0$. Since the track is frictionless, there is no vertical component of tension at $x = 0$. Therefore, Newton's law applied to the segment $(0, h)$ of the string yields

$$\kappa u_x(h, t) = \text{Mass} \ \times \ \text{Acceleration} \approx \rho h u_{tt}(0, t),$$

where $\rho > 0$ is the density of the string. Taking the limit as $h \to 0^+$ we find that $\kappa u_x(0, t) = 0$, as before.

Finally, suppose the ends of the string are free to move along a track, as before, but are now attached to a coiled spring or rubber band obeying Hooke's law. The spring acts to pull the ends of the string back to the equilibrium position. Recall that Hooke's law states that a spring exerts a restoring force proportional to the displacement of the spring from its equilibrium position. If we again consider the segment of string from $x = 0$ to $x = h$, and suppose the equilibrium position of the spring is $u = a$, then we have

$$\kappa u_x(h, t) - K(u(0, t) - a) = \rho h u_{tt}(0, t),$$

where $K > 0$ is the spring constant. Sending $h \to 0^+$ we recover the Robin condition

$$u(0, t) - A u_x(0, t) = a,$$

where $A = \kappa/K > 0$. The computation is similar at the end point $x = l$, and we recover the other Robin condition (3).

1.4 Boundary conditions summary

In summary, three common boundary conditions are

1. **Dirichlet condition:** The value of u is specified on the boundary of the domain ∂D

   $$u(x, y, z, t) = g(x, y, z, t) \quad \text{for all } (x, y, z) \in \partial D \text{ and } t \geq 0,$$

   where g is a given function. When $g = 0$ we have homogeneous Dirichlet conditions.

2. **Neumann condition:** The normal derivative $\partial u/\partial \mathbf{n} = \nabla u \cdot \mathbf{n}$ is specified on the boundary ∂D of the domain, i.e.,

   $$\frac{\partial u}{\partial \mathbf{n}}(x, y, z, t) = g(x, y, z, t) \quad \text{for all } (x, y, z) \in \partial D \text{ and } t \geq 0.$$

   When $g = 0$ we have homogeneous Neumann conditions.

3. **Robin condition:** The Robin condition is a combination of Dirichlet and Neumann conditions, where

$$u(x,y,z,t) + A(x,y,z,t)\frac{\partial u}{\partial \mathbf{n}}(x,y,z,t) = g(x,y,z,t)$$

is specified on the boundary, for given functions A and g.

Example 1.3. The one dimensional heat equation for a rod of length $l > 0$ with homogeneous Dirichlet boundary conditions is

$$\begin{cases} u_t - ku_{xx} = 0, & \text{for } 0 < x < l, t > 0 \\ u(0,t) = u(l,t) = 0, & \text{for } t \geq 0 \\ u(x,0) = f(x), & \text{for } 0 < x < l, \end{cases}$$

where $f(x)$ is a given function. We expect there to be a unique solution $u(x,t)$ of the PDE subject to these initial and boundary conditions, and will prove this later in the course.

Example 1.4. The one dimensional wave equation for a string of length $l > 0$ with homogeneous Neumann boundary conditions is

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & \text{for } 0 < x < l, t > 0 \\ u_x(0,t) = u_x(l,t) = 0, & \text{for } t \geq 0 \\ u(x,0) = f(x), & \text{for } 0 < x < l, \\ u_t(x,0) = g(x), & \text{for } 0 < x < l, \end{cases}$$

where $f(x)$ and $g(x)$ are given functions. Again, we expect there to be a unique solution $u(x,t)$ of the PDE subject to these initial and boundary conditions, and will prove this later in the course.

Well-posed problems

We expect that the physically correct boundary/initial conditions should determine exactly one unique solution of the PDE. Whether or not this is true can be determined mathematically, and this is one question we shall address in this course. We say a PDE in a domain with boundary/initial conditions is *well-posed* if it enjoys the properties

(i) **Existence:** There exists at least one solution of the PDE satisfying all conditions.

(ii) **Uniqueness:** There is at most one solution.

(iii) **Stability:** The unique solution $u(x,t)$ depends in a stable manner on the boundary conditions, initial conditions, and any other data in the problem. This means that if the data are changed by a small amount, the solution u changes by a correspondingly small amount.

We say the equation is *ill-posed* if any of the conditions above are violated.

Stability is a very natural physical assumption. For example, if we consider the Navier-Stokes equations with zero initial conditions (i.e., the water is initially still), then the solution should remain perfectly still for all time. If we change the problem by very lightly stirring the water at time $t = 0$, we should expect the water to move around a tiny bit and come back to being still. That is, the solution should be close to what it was with zero initial conditions. It should not start violently rocking back and forth or splash out of the container. It is interesting to note that the well-posedness of the Navier-Stokes equations is an open problem with a prize of one million dollars from the Clay Mathematics Institute for the first person to settle the question (i.e., either show the PDE is well-posed or is ill-posed).

More concretely, in Example 1.3 the data for the problem are f, and the value of 0 for the Dirichlet boundary conditions at $x = 0$ and $x = l$. The stability condition is satisfied if when f is changed by a small amount, the solution u changes by a similarly small amount, and vice versa for the boundary conditions. We will make this more precise later in the course. One can show, for example, that if u_1 and u_2 are solutions of the heat equation in Example 1.3 with initial conditions f_1 and f_2 at $t = 0$, respectively, then

$$|u_1(x,t) - u_2(x,t)| \leq \max_{0 \leq x \leq l} |f_1(x) - f_2(x)|.$$

Thus, if f_1 and f_2 are uniformly close together then so are u_1 and u_2 for all future times. This is an example of a stability result for this PDE. Whenever we study PDE in this course, we will make a point of checking well-posedness when possible.

Example 1.5. An example of an ill-posed equation is the reverse heat equation

$$u_t + u_{xx} = 0 \quad \text{for } t > 0,$$

subject to an initial condition $u(x,0) = f(x)$. Existence fails for all but very special initial data f (infinitely differentiable f), and stability always fails. To see this, note that the functions

$$u_n(x,t) = \frac{1}{n} e^{n^2 t} \cos(nx),$$

satisfy the reverse heat equation, and the limits

$$\lim_{n\to\infty} u_n(x,0) = 0 \quad \text{and} \quad \lim_{n\to\infty} u_n(0,1) = \lim_{n\to\infty} \frac{1}{n} e^{n^2} = \infty$$

hold. In other words, when n is large, the initial condition $u_n(x,0)$ is very close to zero, while the solution at time $t = 1$ is very far from zero. In fact, the solution grows exponentially fast in t. If u is any solution of the reverse heat equation, then $v = u + u_n$ is also a solution (by linearity). When n is large, the solutions v and u are very close together at time $t = 0$, and very far apart by time $t = 1$. So small perturbations in the initial conditions can yield arbitrarily large perturbations in the solution. This violates stability in a very serious manner.