Math 5587 – Homework 7 (Due Thursday Oct 27)

1. Consider the full Fourier series on the interval $-\ell \le x \le \ell$ for a function $f(x)$

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{\ell}\right) + B_n \sin\left(\frac{n\pi x}{\ell}\right),$$

where

$$A_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)\,dx \quad \text{and} \quad B_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)\,dx.$$

(a) Assume that f is an even function. Show that

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{\ell}\right),$$

where

$$A_n = \frac{2}{\ell}\int_{0}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)\,dx.$$

(b) Assume that f is an odd function. Show that

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{\ell}\right),$$

where

$$B_n = \frac{2}{\ell}\int_{0}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)\,dx.$$

2. Let f be a 2π-periodic k-times continuously differentiable function and consider the complex version of the Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{where} \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx.$$

Recall that for complex-valued functions f and g on the interval $[-\pi,\pi]$ we define

$$(f,g) = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx \quad \text{and} \quad \|f\| = (f,f)^{\frac{1}{2}}.$$

(a) Let g be any 2π-periodic complex-valued differentiable function. Use integration by parts to show that

$$(f,g') = -(f',g)$$

[Hint: Differentiation and complex conjugation commute, so $\frac{d}{dx}\left(\overline{g(x)}\right) = \overline{g'(x)}$.]

(b) Use part (a) to show that

$$c_n = \frac{1}{2\pi(in)^k}\left(f^{(k)}, e^{inx}\right),$$

where $f^{(k)}$ denotes the k^{th} derivative of f. [Hint: Start by using part (a) to show that

$$c_n = \frac{1}{2\pi in}\left(f', e^{inx}\right).$$

Then repeat the argument k times (or use induction).]

(c) Use part (b) and the Cauchy-Schwarz inequality (HW 6) to show that for any $k \geq 1$

$$|c_n| \leq \frac{L_k}{n^k} \quad \text{where} \quad L_k = \frac{1}{\sqrt{2\pi}}\|f^{(k)}\|.$$

That is, if f is k-times continuously differentiable then the Fourier coefficients c_n of f decay to zero at least as fast as $1/n^k$.

3. Let $X(x)$ be a solution of the eigenvalue problem

$$X'' + \lambda X(x) = 0 \quad \text{for } a \leq x \leq b,$$

where λ is a real number.

(a) Assume that

$$XX'\Big|_a^b = X(b)X'(b) - X(a)X'(a) \leq 0. \tag{1}$$

Show that if X is not the zero function, then $\lambda \geq 0$. Hence (1) guarantees there are no negative eigenvalues. [Hint: Show that $\lambda\|X\|^2 = \lambda(X, X) = -(X'', X)$. Then integrate by parts and use (1) to show that $\lambda\|X\|^2 \geq 0$.]

(b) Show that (1) holds when X satisfies

i. Homogeneous Dirichlet boundary conditions $X(a) = X(b) = 0$,
ii. Homogeneous Neumann boundary conditions $X'(a) = X'(b) = 0$,
iii. Mixed boundary conditions $X'(a) = X(b) = 0$ or $X(a) = X'(b) = 0$,
iv. Periodic boundary conditions $X(a) = X(b)$ and $X'(a) = X'(b)$.

4. Let $X_1(x)$ and $X_2(x)$ be solutions of the eigenvalue problems

$$X_1'' + \lambda_1 X_1(x) = 0 \quad \text{for } a \leq x \leq b,$$

and

$$X_2'' + \lambda_2 X_2(x) = 0 \quad \text{for } a \leq x \leq b,$$

for different eigenvalues $\lambda_1 \neq \lambda_2$.

(a) Show that

$$-X_1''X_2 + X_1X_2'' = \frac{d}{dx}(-X_1'X_2 + X_1X_2').$$

[Hint: Use product rule to expand the right hand side.]

(b) Use part (a) and the fundamental theorem of calculus to show that

$$(\lambda_1 - \lambda_2)(X_1, X_2) = (-X_1' X_2 + X_1 X_2')\Big|_a^b.$$

(c) Conclude from part (b) that X_1 and X_2 are orthogonal (i.e., $(X_1, X_2) = 0$) if and only if

$$(-X_1' X_2 + X_1 X_2')\Big|_a^b = 0. \tag{2}$$

(d) Boundary conditions are called **symmetric** whenever (2) holds. By part (b), whenever we have symmetric boundary conditions, eigenfunctions corresponding to different eigenvalues are always orthogonal. Show that each of the boundary conditions listed below are symmetric.

i. Homogeneous Dirichlet boundary conditions $X_i(a) = X_i(b) = 0$,
ii. Homogeneous Neumann boundary conditions $X_i'(a) = X_i'(b) = 0$,
iii. Periodic boundary conditions $X_i(a) = X_i(b)$ and $X_i'(a) = X_i'(b)$,
iv. Robin boundary conditions $X_i(a) - AX_i'(a) = X_i(b) + AX_i'(b) = 0$, where A is a real number.

In each case, the boundary conditions are assumed to hold for both $i = 1$ and $i = 2$.

5. Solve the heat equation

$$\begin{cases} u_t - u_{xx} = 0, & \text{for } 0 < x < \pi \text{ and } t > 0 \\ u(0,t) = 0, & \text{for } t > 0 \\ u(\pi,t) = \pi, & \text{for } t > 0 \\ u(x,0) = \cos(x), & \text{for } 0 \le x \le \pi. \end{cases}$$

Give a complete answer, including the values of the coefficients B_n of the series. After solving for u, compute

$$\lim_{t \to \infty} u(x,t).$$

[Hint: First define $v(x,t) = u(x,t) - x$ to convert the problem into one with homogenous Dirichlet conditions $v(0,t) = v(\pi,t) = 0$. Then solve for v instead.]