MATH 5587 – HOMEWORK 4 (DUE THURSDAY SEPT 29)

1. For a solution $u(x,t)$ of the wave equation

$$u_{tt} - u_{xx} = 0,$$

the energy density is defined as $e(x,t) = (u_t^2 + u_x^2)/2$ and the momentum density is $p(x,t) = u_t u_x$.

   (a) Show that $e_t = p_x$ and $p_t = e_x$.

   (b) Show that both e and p also satisfy the wave equation.

2. Let $u(x,t)$ and $v(x,t)$ be functions such that

$$u_t - ku_{xx} \le v_t - kv_{xx}$$

on the rectangular strip $U_T = (a,b)\times(0,T]$. Prove the following **comparison principle**:

$$\text{If } u \le v \text{ on } \Gamma \text{ then } u \le v \text{ everywhere in } U_T.$$

Recall Γ is the parabolic boundary of U_T, i.e., the sides $x = a$ and $x = b$, and base $t = 0$. [Hint: Apply the maximum principle to $w := u - v$.]

3. Consider the nonlinear heat equation

$$u_t - ku_{xx} + bu_x^2 = 0 \quad \text{for} \quad -\infty < x < \infty \text{ and } t > 0,$$

subject to an initial condition $u(x,0) = f(x)$. This type of PDE arises in stochastic optimal control theory. In this question you will derive a representation formula for the solution $u(x,t)$.

   (a) Define the Cole-Hopf transformation $w(x,t) = e^{-\frac{b}{k}u(x,t)}$. Show that w is a solution of the linear heat equation

$$w_t - kw_{xx} = 0.$$

   (b) Use the fundamental solution of the heat equation to solve for $w(x,t)$.

   (c) Invert the Cole-Hopf transformation to find a formula for u.

4. Consider the heat equation

$$u_t - ku_{xx} = 0 \quad \text{for } 0 < x < \ell \text{ and } t > 0,$$

subject to homogeneous Dirichlet boundary conditions

$$u(0,t) = 0 = u(\ell,t) \quad \text{for } t > 0,$$

and initial condition

$$u(x,0) = f(x) \quad \text{for } 0 < x < \ell.$$

We assume that f is **nonnegative**, that is, $f(x) \ge 0$ for all x.

(a) Use the comparison principle (Problem 2), or the maximum principle, to show that $u(x,t) \geq 0$ for all x and t.

(b) Show that $u_x(0,t) \geq 0$ and $u_x(\ell,t) \leq 0$ for all t. [Hint: Use the definition of these partial derivatives and (a).]

(c) Show that the total heat

$$H(t) = \int_0^\ell u(x,t)\,dx$$

is decreasing in t. That is, show that $H'(t) \leq 0$. Give a short explanation of why heat is decreasing and not conserved.

(d) Show that for each $0 < x < \ell$

$$\lim_{t\to\infty} u(x,t) = 0.$$

That is, all of the heat in the rod eventually dissipates. [Hint: Define

$$v(x,t) := \Phi(x,t+1) = \frac{1}{\sqrt{4\pi k(t+1)}} \exp\left(-\frac{x^2}{4k(t+1)}\right),$$

where Φ is the fundamental solution of the heat equation. Recall that v satisfies the heat equation $v_t - kv_{xx} = 0$ for $t > -1$. Explain how to use the comparison principle from problem 2 to show that $u \leq Cv$ for an appropriate constant C depending on f and k. Complete the proof from here.]

5. **Maximum principle:** Consider the heat equation

$$\text{(H)} \quad \begin{cases} u_t - u_{xx} = 0, & -\infty < x < \infty,\ t > 0 \\ u(x,0) = \varphi(x), & -\infty < x < \infty. \end{cases}$$

As it turns out, there are infinitely many solutions u of the above heat equation. All but one solution are “non-physical” and grow exponentially fast as $x \to \pm\infty$. In this question, you will show that if φ is bounded, then there is a unique bounded solution $u(x,t)$. The proof involves the maximum principle that we dicussed in class for bounded domains.

Throughout the question let u be a bounded solution of (H); this means there exists $C > 0$ such that $|u(x,t)| \leq C$ for all (x,t).

(a) Show that $w(x,t) = x^2 + 2t$ solves the heat equation $w_t = w_{xx}$.

(b) For every $\varepsilon > 0$ show that

$$u(x,t) \leq \varepsilon w(x,t) + M \quad \text{for all } x \in \mathbb{R} \text{ and } t > 0,$$

where $M > 0$ is any number satisfying $\varphi(x) \leq M$ for all $x \in \mathbb{R}$. [Hint: For $N > 0$ let R_N denote the rectangle

$$R_N = [-N,N] \times [0,N] = \{(x,t) : -N \leq x \leq N \text{ and } 0 \leq t \leq N\}.$$

Show that there exists $\overline{N} > 0$ such that for all $N > \overline{N}$, $u \leq \varepsilon w + M$ on the sides $x = -N$ and $x = N$, and base $t = 0$ of R_N. Then apply the comparison principle from Problem 2.]

(c) Let $M > 0$ such that $\varphi(x) \le M$ for all $x \in \mathbb{R}$. Show that $u \le M$.

(d) Show that there is at most one bounded solution u of (H) when φ is bounded. [Hint: Take two bounded solutions u, v and consider $w := u - v$. Then choose $M = 0$ to show that $u \le v$. Complete the proof from here.]

It is possible to prove a stronger result; namely that there is at most one solution u of (H) satisfying the exponential growth estimate

$$u(x,t) \le Ae^{ax^2}$$

for constants $A > 0$ and $a > 0$. The proof is similar to this exercise, except that w has a different form (similar to HW3 #5). This means that the "non-physical" solutions all grow faster than Ae^{ax^2} as $x \to \pm\infty$.

6. Consider the heat equation on the half line

$$u_t - ku_{xx} = 0 \quad \text{for} \quad x > 0 \text{ and } t > 0,$$

with homogeneous Neumann boundary conditions $u_x(0,t) = 0$ for all $t > 0$, and intial condition $u(x,0) = f(x)$ for $x > 0$. Use the method of even extensions, as outlined in the notes, to show that the solution $u(x,t)$ is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left(e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt}\right) f(y)\, dy.$$