MATH 5587 – HOMEWORK 3 (DUE THURSDAY SEPT 22)

1. Solve $u_{tt} = c^2 u_{xx}$ with $u(x,0) = x^2$ and $u_t(x,0) = \cos(x)$.

2. Find the solution of $u_{xx} - 3u_{xt} - 4u_{tt} = 0$ with $u(x,0) = e^x$ and $u_t(x,0) = 0$. [Hint: Factor the operator into the form $(\partial_x - 4\partial_t)(\partial_x + \partial_t)u = 0$, and proceed as we did in class to derive d'Alembert's formula.]

3. **The hammer blow:** Consider the wave equation $u_{tt} = u_{xx}$ on the entire real line $-\infty < x < \infty$ with zero initial position $u(x,0) = 0$ and initial velocity $u_t(x,0) = g(x)$, where $g(x) = 1$ for $|x| < 1$ and $g(x) = 0$ for $|x| \geq 1$. Sketch the solution at time instants $t = 1/2, 1, 3/2, 2$ and $t = 5/2$. What is the maximum displacement $\max_x u(x,t)$?

4. Suppose that $u(x,t)$ solves the wave equation $u_{tt} = u_{xx}$ for $0 < x < \ell$ and $t \geq 0$ with Robin boundary conditions

$$u(0,t) - u_x(0,t) = 0 \quad \text{and} \quad u(\ell,t) + u_x(\ell,t) = 0.$$

   (a) Show that the energy

$$E(t) = \frac{1}{2}\int_0^\ell u_t(x,t)^2 + u_x(x,t)^2\,dx + \frac{1}{2}u(0,t)^2 + \frac{1}{2}u(\ell,t)^2$$

   is conserved (i.e., $E'(t) = 0$).

   (b) Give a physical explanation for the additional two terms in $E(t)$. [Recall the physical explanation of the Robin conditions for the wave equation from lecture 2.]

5. 'Blow-up' for the heat equation.

   (a) Show that the function

$$u(x,t) = \frac{1}{\sqrt{1-t}}\exp\left(\frac{x^2}{4(1-t)}\right)$$

   is a solution of the heat equation

$$u_t - u_{xx} = 0 \quad \text{for } -\infty < x < \infty \text{ and } 0 < t < 1.$$

   (b) Sketch the functions $t \mapsto u(0,t)$ and $x \mapsto u(x,1/2)$.

   (c) Can you give a physical explanation for the 'blow-up' observed at $t = 1$?

6. (a) Solve the diffusion equation $u_t = u_{xx}$ on the real line $-\infty < x < \infty$ with initial condition $u(x,0) = x^2$ **without** using the fundamental solution. [Hint: Note that u_{xxx} satisfies the same diffusion equation with zero initial condition. Therefore $u_{xxx} \equiv 0$ and we find that $u(x,t) = A(t)x^2 + B(t)x + C(t)$. Solve for $A(t)$, $B(t)$ and $C(t)$.]

   (b) Use your answer to part (a) to deduce the value of

$$\int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx.$$

   [Hint: Write an expression for $u(0,1)$ using the fundamental solution and make a substitution in the integral.]