Math 5587 Final Exam Information

- The final exam will take place on Tuesday, December 20 4:45pm-6:45pm in Vincent Hall 207. Please arrive early so we can start on time.
- The exam can potentially cover anything in the course, with the exception of the lectures on the eikonal equation and maze navigation.
- The exam is closed book. No textbooks, notes, or calculators are allowed. The formula sheet below will be provided on the exam.
- The exam will have 8 questions, ranging in levels of difficulty. It is a good idea to look through the questions first and complete the ones your are most comfortable with early in the exam. Below are a collection of sample questions from the last third of the course. Please see the sample problems for midterms 1 and 2 for practice problems from the first two thirds of the semester.

Sample questions

1. Determine whether the following statements are true or false. No justification is required.
   (a) The Green's function is the same for every domain D.
   (b) A finite difference scheme is stable if all eigenvalues obtained via the Von-Neumann analysis have modulus at most 1.
   (c) A finite difference scheme is unstable when all eigenvalues obtained via the Von-Neumann analysis are negative.
   (d) The weak maximum principle can be used to prove uniqueness of solutions to Poisson's equation.
   (e) The function $u(x,y) = xy(1-x)(1-y)e^{\cos(xy)\sin(x+y)xy}$ is harmonic in the rectangle $0 < x < 1$ and $0 < y < 1$.
   (f) Energy methods can only be used to prove uniqueness for solutions of linear partial differential equations.
2. State the strong and weak maximum principles for harmonic functions.
3. Use energy methods to prove uniqueness of solutions to the PDE
$$-\Delta u + u^3 = f \ \text{ in } D$$
   with $u = 0$ on ∂D. [Hint: Write $w = u - v$ and use the identity
$$u^3 - v^3 = (u-v)(u^2 + uv + v^2).$$
   ]
4. Use the maximum principle to prove uniqueness of solutions of the PDE
$$-\Delta u + u^3 = f \ \text{ in } D$$
   with $u = 0$ on ∂D. [Hint: Proceed similarly to previous problem, but consider the maximum of $w = u - v$.]

5. Compute the integral
$$\iint_B \log(x^2+y^2)\,dxdy,$$
where B is the ball of radius 1 centered at $(2,0)$. [Hint: Use the mean value property.]

6. Consider the PDE
$$u_t - u_{xx} + u_x = 0.$$
Find a stable finite difference numerical scheme and find the CFL condition for stability.

7. Consider the reverse heat equation.
$$u_t + u_{xx} = 0.$$
Show that the scheme using forward differences in t and centered in x is always unstable. Give a brief explanation of why.

8. The *convolution* of two functions $\varphi(x)$ and $\psi(x)$ is the function $\varphi * \psi$ defined by
$$(\varphi * \psi)(x) := \int_{-\infty}^{\infty} \varphi(x-y)\psi(y)\,dy.$$
   (a) Show that $\varphi * \psi = \psi * \varphi$.
   (b) For a continuous function f and test functions φ and ψ show that
$$\langle \psi * f, \varphi\rangle = \int_{-\infty}^{\infty} (\psi * f)(x)\varphi(x)\,dx = \int_{-\infty}^{\infty} f(x)(\widetilde{\psi} * \varphi)(x)\,dx = \langle f, \widetilde{\psi} * \varphi\rangle,$$
   where $\widetilde{\psi}(x) := \psi(-x)$.
   (c) In light of (b), we define the convolution of a generalized function f with a smooth function ψ to be the generalized function
$$\langle \psi * f, \varphi\rangle := \langle f, \widetilde{\psi} * \varphi\rangle,$$
   Show that $\psi * \delta = \psi$.

9. State the properties defining the Green's function for a domain $D \subseteq \mathbb{R}^3$.

10. Show that the Green's function for a bounded domain $D \subseteq \mathbb{R}^3$ is unique.

Formula Sheet

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nx) + B_n \sin(nx) = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = \begin{cases} \pi, & \text{if } n = m \\ 0, & \text{otherwise.} \end{cases}.$$

$$\int_{-\pi}^{\pi} \cos(nx)\sin(mx)\,dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} e^{inx}e^{-imx}\,dx = \begin{cases} 2\pi, & \text{if } n = m \\ 0, & \text{otherwise.} \end{cases}.$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)^2\,dx = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = \frac{1}{2}A_0^2 + \sum_{n=1}^{\infty} A_n^2 + B_n^2.$$

$$\iiint_D \nabla u \cdot \nabla v\,d\mathbf{x} = \iint_{\partial D} u\frac{\partial v}{\partial \mathbf{n}}\,dS - \iiint_D u\Delta v\,d\mathbf{x}.$$

$$\iiint_D u\Delta v - v\Delta u\,d\mathbf{x} = \iint_{\partial D} u\frac{\partial v}{\partial \mathbf{n}} - v\frac{\partial u}{\partial \mathbf{n}}\,dS.$$

$$\iiint_D \Delta u\,d\mathbf{x} = \iint_{\partial D} \frac{\partial u}{\partial \mathbf{n}}\,dS.$$