Math 5587 Final Exam

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Name:________________________________

Instructions:

1. I recommend looking over the problems first and starting with those you feel most comfortable with.
2. Unless otherwise noted, be sure to include explanations to justify each step in your arguments and computations. For example, make sure to check that the hypotheses of any theorems you use are satisfied.
3. All work should be done in the space provided in this exam booklet. Cross out any work you do not wish to be considered. Additional white paper is available if needed.
4. Books, notes, calculators, cell phones, pagers, or other similar devices are not allowed during the exam. Please turn off cell phones for the duration of the exam. You may use the formula sheet attached to this exam.
5. If you complete the exam within the last 15 minutes, please remain in your seat until the examination period is over.
6. In the event that it is necessary to leave the room during the exam (e.g., fire alarm), this exam and all your work must remain in the room, face down on your desk.

Problem	Score
1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
7	/10
8	/10
Total:	/80

1. Determine whether each statement is true or false. No justification required. [10 points]

   (a) Solutions of the wave equation obey the maximum principle.

   (b) Solutions of the heat equation always conserve the total heat, no matter what the boundary conditions are.

   (c) Consistent finite difference schemes are always stable.

   (d) The Fourier series for any 2π-periodic function always converges uniformly.

   (e) The method of characteristics can be used to solve the heat equation.

2. Answer each question below. No justification is required. [10 points]

   (a) Give an example of a real-valued function $f(x)$ that is not generalized function.

   (b) Give an example of a generalized function that cannot be interpreted as an ordinary real-valued function.

   (c) Give an example of linear PDE and a nonlinear PDE.

   (d) Give an example of a function whose full Fourier series on $-\pi \leq x \leq \pi$ converges uniformly.

   (e) Give an example of a PDE on a domain with boundary conditions that is not well-posed.

3. Solve the partial differential equation $u_t + xu_x = 0$ on the domain $t > 0$ and $-\infty < x < \infty$ with initial condition $u(x, 0) = x^2$. [10 points]

4. Solve the partial differential equation $u_{tt} - u_{xx} = 0$ on the domain $t > 0$ and $-\infty < x < \infty$ with initial conditions $u(x, 0) = \sin(x)$ and $u_t(x, 0) = -2xe^{-x^2}$. [10 points]

5. (a) State either form of the mean-value property for harmonic functions in $\mathbb{R}^2$. [5 points]

(b) Use the mean value property to compute the integral

$$\iint_B x^2 - y^2 \, dxdy,$$

where B is the ball of radius $r > 0$ centered at (x_0, y_0). Your answer should depend on r, x_0 and y_0. [5 points]

6. Let

$$f_n(x) = n^2 \left(\delta\left(x + \tfrac{1}{n}\right) - 2\delta(x) + \delta\left(x - \tfrac{1}{n}\right)\right).$$

Show that $f_n(x) \rightharpoonup \delta''(x)$ weakly as $n \to \infty$. [10 points]

7. Let $D \subset \mathbb{R}^2$ be open and bounded, and suppose that u is a smooth solution of the partial differential equation

$$u + u_x^2 + u_y^2 = 1 \quad \text{in } D$$

and $u = 0$ on ∂D. Show that $0 \leq u \leq 1$ everywhere in D. [10 points] [Hint: Use maximum principle arguments. By this, I mean examine the points where u attains its maximum and minimum values...]

8. Let $D \subset \mathbb{R}^2$ be open and bounded. Use energy methods to prove uniqueness of solutions of the partial differential equation

$$\begin{cases} u - \Delta u + u_x + u_y = f & \text{in } D, \\ u = g & \text{on } \partial D. \end{cases}$$

[10 points] [Hint: You will need to use Cauchy's inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ twice in your argument.]

Scratch paper

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Formula Sheet

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nx) + B_n \sin(nx) = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = \begin{cases} \pi, & \text{if } n = m \\ 0, & \text{otherwise.} \end{cases}.$$

$$\int_{-\pi}^{\pi} \cos(nx)\sin(mx)\,dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} e^{inx}e^{-imx}\,dx = \begin{cases} 2\pi, & \text{if } n = m \\ 0, & \text{otherwise.} \end{cases}.$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)^2\,dx = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = \frac{1}{2}A_0^2 + \sum_{n=1}^{\infty} A_n^2 + B_n^2.$$

$$\iiint_D \nabla u \cdot \nabla v\,d\mathbf{x} = \iint_{\partial D} u\frac{\partial v}{\partial \mathbf{n}}\,dS - \iiint_D u\Delta v\,d\mathbf{x}.$$

$$\iiint_D u\Delta v - v\Delta u\,d\mathbf{x} = \iint_{\partial D} u\frac{\partial v}{\partial \mathbf{n}} - v\frac{\partial u}{\partial \mathbf{n}}\,dS.$$

$$\iiint_D \Delta u\,d\mathbf{x} = \iint_{\partial D} \frac{\partial u}{\partial \mathbf{n}}\,dS.$$