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Exceptional regular singular points of second-order ODEs

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Frobenius' method for solving

$$u'' + \frac{b(x)}{x}\,u' + \frac{c(x)}{x^2}\,u \;=\; 0 \qquad\qquad \text{(with } b, c \text{ analytic near 0)}$$

is slightly more complicated when the indicial equation

$$\alpha(\alpha-1) + b(0)\alpha + c(0) \;=\; 0$$

has *repeated roots* or *roots differing by an integer*. [1] An important example in which this occurs is the differential equation in radial coordinates for spherical functions on hyperbolic n-space:

$$u'' + (n-1)\coth r \cdot u' - \lambda\, u \;=\; 0$$

The point 0 is a regular singular point for this equation. The indical equation is

$$\alpha(\alpha-1) + (n-1)\alpha \;=\; 0$$

When $n = 2$, the indicial equation has a double root 0. When $n \geq 3$, the roots are 0 and $2 - n$, which differ by an integer.

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1. *Solving second-order ODEs*

[1.1] Frobenius' method

Let

$$E \;=\; \frac{d^2}{dx^2} + \frac{b(x)}{x}\,\frac{d}{dx} + \frac{c(x)}{x^2} \qquad\qquad \text{(with } b, c \in \mathbb{C}[[x]])$$

be the differential operator and let

$$p(\alpha) \;=\; \alpha(\alpha-1) + b(0)\alpha + c(0) \;\;\in\;\; \mathbb{C}[\alpha]$$

Consider

$$f(x,\alpha) \;=\; x^\alpha \cdot \sum_{n\geq 0} a_n\, x^n \qquad\qquad \text{(on } x > 0 \text{, with } a_0 \neq 0)$$

with a_n depending on α. We can solve

$$Ef(x,\alpha) \;=\; p(\alpha)\cdot a_0 \cdot x^\alpha$$

by recursively solving for the coefficients a_n, as follows. Expand

[1] For example, see [Coddington 1961], chapter 4, section 6.

$$Ef(x,\alpha) \;=\; p(\alpha)\cdot a_0 \cdot x^\alpha \;+\; \Big[p(\alpha+1)\cdot a_1 + (\text{something involving } a_0)\Big]\cdot x^{\alpha+1}$$
$$+ \Big[p(\alpha+2)\cdot a_2 + (\text{something involving } a_0, a_1)\Big]\cdot x^{\alpha+2} \;+\; \ldots$$

That is, the coefficient of $x^{\alpha+n-2}$ in Ef is

$$\Big[(\alpha+n)(\alpha+n-1) + b(0)(\alpha+n) + c(0)\Big]\cdot a_n \;+\; (\text{linear combination of } a_\ell\text{'s with } \ell < n)$$

The coefficient of x^α is $p(\alpha)\cdot a_0$. For $n \geq 1$, as long as the coefficient $p(\alpha+n) \in \mathbb{C}$ of a_n is non-zero, the condition that the coefficient of $x^{\alpha+n}$ *vanish* determines a_n in terms of a_ℓ with $\ell < n$.

Thus, when the two roots of $p(\alpha) = 0$ do *not* differ by an integer, the differential equation has a solution of the form $x^\alpha \sum_{n\geq 0} a_n x^n$ with *either* root α, since the recursion for the coefficients succeeds. It is not difficult to prove that the series has positive radius of convergence: we prove this later.

[1.1.1] Remark: In any case, a *first solution* can be obtained by taking α to be the solution of the indicial equation with larger real part, thus avoiding failure of the recursion. The problem is to obtain a *second solution* when the roots of the indicial equation are equal or differ by an integer.

[1.2] Simpler exceptional case

The simpler *exceptional case* is that $p(\alpha) = (\alpha - \alpha_o)^2$. Then the recursion for the coefficients a_n succeeds, but produces just one solution $f(x, \alpha_o)$. To obtain a second solution, apply $\partial/\partial\alpha$ to

$$Ef(x,\alpha) \;=\; p(\alpha)\cdot a_0 \cdot x^\alpha$$

and evaluate at $\alpha = \alpha_o$:

$$E\frac{\partial f}{\partial\alpha}(x,\alpha_o) \;=\; p'(\alpha_o)\, a_0\, x^{\alpha_o} + p(\alpha_o)\, a_0\, x^{\alpha_o}\log x \;=\; 0$$

Given the expansion $f(x,\alpha) = x^\alpha \sum_n a_n\, x^n$, the expansion of the second solution is

$$\frac{\partial f}{\partial\alpha}(x,\alpha_o) \;=\; x^{\alpha_o}\sum_n \frac{\partial a_n}{\partial\alpha}\, x^n \;+\; \log x\cdot x^{\alpha_o}\sum_n a_n\, x^n \;=\; x^{\alpha_o}\sum_n \frac{\partial a_n}{\partial\alpha}\, x^n + \log x\cdot f(x,\alpha_o)$$

Since a_0 does not depend on α, the differentiation in α annihilates this term. Thus, the leading term of the second solution is $\log x\cdot x^\alpha\cdot a_0$: taking $a_0 = 1$,

$$\begin{cases} (\text{first solution}) & = \; x^{\alpha_o} + \ldots \\ (\text{second solution}) & = \; \log x\cdot x^{\alpha_o} + \ldots \end{cases} \qquad (\text{for } p(\alpha) = (\alpha-\alpha_o)^2)$$

We prove convergence later.

[1.3] Less-simple exceptional case

In the less-simple exceptional case, the roots of the indicial equation are α_o and $\alpha_o - n_o$ with a positive integer n_o. The recursion for the coefficients of a solution of the form $x^\alpha \sum_{n\geq 0} a_n x^n$ may fail at n_o, because it apparently requires division by

$$p((\alpha_o - n_o) + n_o) \;=\; p(\alpha_o) \;=\; 0$$

That is, $f(x,\alpha)$ is of the form

$$f(x,\alpha) \;=\; A(x,\alpha) + \frac{B(x,\alpha)}{\alpha - (\alpha_o - n_o)}$$

where $A(x,\alpha)$ is the polynomial in x consisting of terms of degree below n_o, and $B(x,\alpha)$ is the part above-or-equal that degree. Application of the differential operator E to $(\alpha - (\alpha_o - n_o)) \cdot f(x,\alpha)$ gives

$$E\Big(\frac{\partial}{\partial\alpha}\Big|_{\alpha=\alpha_o-n_o}\Big((\alpha-(\alpha_o-n_o))\cdot f(x,\alpha)\Big)\Big) \;=\; \frac{\partial}{\partial\alpha}\Big|_{\alpha=\alpha_o-n_o}\Big((\alpha-(\alpha_o-n_o))\cdot p(\alpha)\, a_0\, x^\alpha\Big) \;=\; 0$$

That is, a *second solution* is

$$\frac{\partial}{\partial\alpha}\Big|_{\alpha=\alpha_o-n_o}\Big((\alpha-(\alpha_o-n_o))\cdot f(x,\alpha)\Big) \;=\; A(x,\alpha_o-n_o) + \frac{\partial B}{\partial\alpha}(x,\alpha_o-n_o)$$

Term-wise, writing

$$f(x,\alpha) \;=\; x^\alpha \sum_{n<n_o} a_n\, x^n \;+\; \frac{x^\alpha}{\alpha-(\alpha_o-n_o)} \sum_{n\ge n_o} b_n\, x^n$$

a second solution is

$$\frac{\partial}{\partial\alpha}\Big|_{\alpha=\alpha_o-n_o}\Big((\alpha-(\alpha_o-n_o))\cdot f(x,\alpha)\Big) \;=\; x^{\alpha_o-n_o}\sum_{n<n_o} a_n\, x^n \;+\; \log x\cdot x^{\alpha_o-n_o}\sum_{n\ge n_o} b_n\, x^n \;+\; x^{\alpha_o-n_o}\sum_{n\ge n_o}\frac{\partial b_n}{\partial\alpha}\, x^n$$

In particular, the log terms only appear in the higher-order terms: taking $a_0 = 1$,

$$\begin{cases} \text{(first solution)} & = \;\; x^{\alpha_o} + \ldots \\ \text{(second solution)} & = \;\; x^{\alpha_o-n_o} + \ldots \end{cases} \qquad \text{(for } p(\alpha) = (\alpha-\alpha_o)(\alpha-(\alpha_o-n_o)) \text{ and } 0 < n_o \in \mathbb{Z})$$

We prove convergence later.

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2. *Examples*

[2.1] Euclidean n-space: radial eigenfunctions near 0

In radial coordinates, the Laplacian on Euclidean n-space is

$$u'' + \frac{n-1}{r}u'$$

The corresponding eigenvalue equation is

$$u'' + \frac{n-1}{r}u' - \lambda u \;=\; 0$$

The equation for a *fundamental solution* is also of interest:

$$u'' + \frac{n-1}{r}u' - \lambda u \;=\; \delta \qquad\qquad \text{(with Dirac delta)}$$

Rewritten to appraise the regular singular point at 0, the operator is

$$u'' + \frac{n-1}{r}u' - \frac{r^2\lambda}{r^2}u$$

also showing that the indicial equation is independent of λ: it is

$$\alpha(\alpha-1) + (n-1)\alpha \;=\; 0$$

with roots $0, 2-n$, notably differing by an integer. For $n = 2$, the root is double-0. With $\lambda = 0$, the equation

$$u'' + \frac{n-1}{r}u' \;=\; 0$$

is of Euler-Cauchy type, explicitly solvable: the two solutions are

$$\text{first and second solutions} \;=\; \begin{cases} 1 \quad \text{and} \quad r^{2-n} \quad (\text{for } n \neq 2) \\ \\ 1 \quad \text{and} \quad \log r \quad (\text{for } n = 2) \end{cases} \qquad\qquad (\text{for } \lambda = 0)$$

In this example, the *first solution* 1 is *analytic* at 0. The *second solution* is *not* analytic at 0.

For general λ, the first solution is analytic at 0, the second solution is not analytic at 0, and should be expected to have logarithmic terms even for $n > 2$, although these will not appear in the leading term:

$$\text{first and second solutions} \;=\; \begin{cases} 1 + \ldots \quad \text{and} \quad r^{2-n} + \ldots \quad (\text{for } n \neq 2) \\ \\ 1 + \ldots \quad \text{and} \quad \log r + \ldots \quad (\text{for } n = 2) \end{cases} \qquad\qquad (\text{for } \lambda \neq 0)$$

[2.1.1] Remark: Computation of asymptotics at the *irregular* singular point *at infinity* reveals that for *odd* n the two solutions of the homogeneous equation have elementary expressions of the form

$$e^{\pm r\sqrt{\lambda}} \cdot (\text{polynomial in } \frac{1}{r})$$

For example,

$$\begin{cases} e^{\pm r\sqrt{\lambda}} & (\text{on } \mathbb{R}^1) \\ \dfrac{e^{\pm r\sqrt{\lambda}}}{r} & (\text{on } \mathbb{R}^3) \\ e^{\pm r\sqrt{\lambda}}\Big(\dfrac{1}{r^2} - \dfrac{1}{\pm r^3\sqrt{\lambda}}\Big) & (\text{on } \mathbb{R}^5) \\ e^{\pm r\sqrt{\lambda}}\Big(\dfrac{1}{r^3} - \dfrac{3}{\pm r^5\sqrt{\lambda}} + \dfrac{3}{r^7\,\lambda}\Big) & (\text{on } \mathbb{R}^7) \end{cases}$$

Thus, in fact, logarithmic terms *do not appear* in the asymptotic near 0 for n odd, although this is not obvious from the expansion at 0.

[2.1.2] Remark: Further, in the elementary expressions for odd n, for the blow-ups at 0 to cancel, up to a constant the *first solution*, analytic at 0, must be the *sum* of the two expressions. Thus, the first solution can never have exponential decay at infinity. Unless $\sqrt{\lambda}$ is purely imaginary, it will have exponential blow-up. For $\sqrt{\lambda}$ purely imaginary, it will have polynomial decay.

[2.1.3] Remark: Since the support of δ is just $\{0\}$, a solution of

$$u'' + \frac{n-1}{r}u' - \lambda u \;=\; \delta$$

satisfies the *homogeneous* form of the equation *away from* 0, so is a linear combination of the first and second solution, with the second solution being necessary to produce δ at 0. For $\mathrm{Re}(\lambda) > 0$, Fourier transform methods on $\mathbb{R}^n$ produce an exponentially decreasing fundamental solution.

[2.2] Hyperbolic n-space: radial eigenfunctions near 0

The differential equation in radial coordinates for zonal spherical functions on hyperbolic n-space is

$$u'' + (n-1)\coth r \cdot u' - \lambda\, u \;=\; 0$$

Rewritten to highlight the regular singular point at $r = 0$, it is

$$u'' + \frac{(n-1) r \coth r}{r} \cdot u' - \frac{r^2 \lambda}{r^2}\, u \;=\; 0$$

The indical equation is the same as that for Euclidean space, and is independent of λ:

$$\alpha(\alpha-1) + (n-1)\alpha \;=\; 0$$

When $n = 2$, the indicial equation has a double root 0. When $n \geq 3$, the roots are 0 and $2 - n$, which differ by an integer.

Since the solution of the indicial equation with larger real part is 0, the *first solution* is *regular* at $r = 0$. This is the *zonal spherical function* for that eigenvalue.

When $n = 2$, the *second solution* has a logarithmic leading term. For $n > 2$, the second solution has logarithmic terms further out in the expansion, but not in the leading term:

$$\text{first and second solutions} \;=\; \begin{cases} 1 + \ldots \quad \text{and} \quad r^{2-n} + \ldots & (\text{for } n \neq 2) \\ \\ 1 + \ldots \quad \text{and} \quad \log r + \ldots & (\text{for } n = 2) \end{cases} \qquad (\text{near } 0)$$

[2.2.1] Remark: Away from 0, a solution to

$$u'' + (n-1)\coth r \cdot u' - \lambda\, u \;=\; \delta \qquad (\text{Dirac delta at base point})$$

satisfies the corresponding homogeneous equation. Being a linear combination of the first and second solutions, it must always non-trivially include the second solution to obtain δ at 0. In particular, the full asymptotics of a fundamental solution near 0 should be expected to include logarithmic terms for $n > 2$, although not in the leading term.

[2.3] Hyperbolic n-space: special radial eigenfunctions near $+\infty$

Unlike the Euclidean case, in hyperbolic n-space the radial (spherically symmetric) Laplacian has a *regular* singular point at infinity, in suitable coordinates. With $y = e^r$, it suffices to look at $y \to 0^+$ or, equivalently $y \to +\infty$. The y-coordinate is already adapted to the case $y \to 0^+$, so we consider this. The equation becomes

$$y^2 u'' + y\Big(1 + (n-1)\frac{y + 1/y}{y - 1/y}\Big)u' - \lambda u \;=\; 0 \qquad (\text{as } y \longrightarrow 0^+)$$

The singular point at 0 is visibly regular, with indicial equation

$$\alpha(\alpha - (n-1)) - \lambda \;=\; 0$$

Parametrizing the eigenvalue by $\lambda = s(s - (n-1))$, the roots are

$$\alpha \;=\; \begin{cases} s \\ \\ (n-1) - s \end{cases} \qquad (\text{with eigenvalue } \lambda = s(s-(n-1)))$$

The special case $s = (n-1)/2$ plays a central role in HarishChandra's treatment of Schwartz functions, and in that case $(n-1)/2$ is a *double* root. Thus, the two solutions are

$$\begin{cases} \text{(first solution)} & = \quad y^{\frac{n-1}{2}} + \ldots \\ \text{(second solution)} & = \quad \log y \cdot y^{\frac{n-1}{2}} + \ldots \end{cases} \qquad (\text{as } y \longrightarrow 0^+)$$

The spherical symmetry gives symmetry under $y \to 1/y$, so

$$\begin{cases} \text{(first solution)} & = \quad y^{-\frac{n-1}{2}} + \ldots \\ \text{(second solution)} & = \quad \log y \cdot y^{-\frac{n-1}{2}} + \ldots \end{cases} \qquad (\text{as } y \longrightarrow +\infty)$$

In terms of the original radius r,

$$\begin{cases} \text{(first solution)} & = \quad e^{-\frac{n-1}{2}|r|} + \ldots \\ \text{(second solution)} & = \quad |r| \cdot e^{-\frac{n-1}{2}|r|} + \ldots \end{cases} \qquad (\text{as } |r| \longrightarrow +\infty)$$

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3. *Convergence*

[3.1] First solutions

We recall the relatively easy argument for convergence of the series for the *first solution*.

Let $A, M \geq 1$ be large enough so that

$$b(x) \;=\; \sum_{n\geq 0} b_n\, x^n \qquad (\text{with } |b_n| \leq A \cdot M^n)$$

$$c(x) \;=\; \sum_{n\geq 0} c_n\, x^n \qquad (\text{with } |c_n| \leq A \cdot M^n)$$

Inductively, suppose that $|a_\ell| \leq (CM)^\ell$, with a constant $C \geq 1$ to be determined in the following. Then

$$|n(n+\alpha-\alpha')\cdot a_n| \;\leq\; A\sum_{i=1}^n |n-i+\alpha| M^i \cdot (CM)^{n-i} + A\sum_{i=1}^n M^i \cdot (CM)^{n-i} \;\leq\; AM^nC^{n-1}\Big(\frac{n(n+1)}{2} + n|\alpha| + n\Big)$$

Dividing through by $n|n+\alpha-\alpha'|$, this is

$$|a_n| \;\leq\; AM^n \cdot C^{n-1}\,\frac{(n+1)+2|\alpha|+2}{2|n+\alpha-\alpha'|}$$

This motivates the choice

$$C \;\geq\; \sup_{1\leq n\in\mathbb{Z}} \frac{(n+1)+2|\alpha|+2}{2|n+\alpha-\alpha'|}$$

which gives $|a_n| \leq A(CM)^n$, and a positive radius of convergence.

[3.2] Simpler second solution

Proof of convergence of the second solution when $p(\alpha) = (\alpha-\alpha_o)^2$ requires finer attention to the dependence on α. The recursion for the coefficients a_n of $f(x,\alpha)$ is

$$-p(\alpha+n)\cdot a_n \;=\; \sum_{0\leq \ell<n} (\alpha+\ell) a_\ell\, b_{n-\ell} \;+\; \sum_{0\leq\ell<n} a_\ell\, c_{n-\ell}$$

Differentiating with respect to α,

$$-p'(\alpha+n)\cdot a_n - p(\alpha+n)\cdot\frac{\partial a_n}{\partial\alpha} \;=\; \sum_{0\le\ell<n} a_\ell\, b_{n-\ell} \;+\; \sum_{0\le\ell<n}(\alpha+\ell)\frac{\partial a_\ell}{\partial\alpha}\, b_{n-\ell} \;+\; \sum_{0\le\ell<n}\frac{\partial a_\ell}{\partial\alpha}\, c_{n-\ell}$$

Invoking the estimates from above, and inductively supposing $|\partial a_n/\partial\alpha| \le (DM)^n$ with $D \ge C$,

$$|p(\alpha+n)\cdot\frac{\partial a_n}{\partial\alpha}| \quad\le\quad |p'(\alpha+n)\cdot(CM)^n|$$

$$+\sum_{0\le\ell<n}(CM)^\ell\, AM^{n-\ell} \;+\sum_{0\le\ell<n}|\alpha+\ell|(DM)^\ell\, AM^{n-\ell} \;+\sum_{0\le\ell<n}(DM)^\ell\, AM^{n-\ell}$$

Thus,

$$|\frac{\partial a_n}{\partial\alpha}| \;\le\; \Big|\frac{p'(\alpha+n)\cdot(CM)^n}{p(\alpha+n)}\Big| \;+\; \frac{A(n+n|\alpha|+\frac{1}{2}n(n+1)+n)\,D^{n-1}M^n}{|p(\alpha+n)|}$$

Taking

$$D \;\ge\; \sup_{n\ge1}\Big(\Big|\frac{p'(\alpha+n)\cdot C}{p(\alpha+n)}\Big| \;+\; \frac{A(3n+n^2+n|\alpha|)}{|p(\alpha+n)|}\Big)$$

gives the inductive step, proving that the second solution converges absolutely on some non-trivial interval $0 < x < x_o$.

[3.3] Less-simple second solution

A similar but messier argument succeeds in the less-simple case, as well, proving convergence on some non-trivial interval.

Bibliography

[Coddington 1961] E.A. Coddington, *An introduction to ordinary differential equations*, Dover, 1961.
