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Traces, Cauchy identity, Schur polynomials

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1. Example: GL_2
2. $GL_n(\mathbb{C})$ and $U(n)$
3. Decomposing holomorphic polynomials over $GL_n \times GL_n$

The unobvious *Cauchy identity*^[1] is proven by taking a *trace* of a representation of $GL_n(\mathbb{C})$ in two different ways: one as the sum of traces of irreducible subrepresentations, the other as sum of weight spaces, that is, irreducible subrepresentations for a maximal torus.

The representation whose trace we take is the space of holomorphic polynomials on n -by- n complex matrices.

The holomorphic irreducibles occurring are identified by restriction to the realform $U(n)$, invoking the decomposition of the biregular representation of $U(n)$.

Such identities arise in Rankin-Selberg integral representations of L -functions. For GL_2 , a naive, direct computation is sufficient. However, for general GL_n and other higher-rank groups, direct computation is inadequate. Further, connecting local Rankin-Selberg computations to Schur functions usefully connects these computations to the Shintani-Casselman-Shalika formulas for spherical p -adic Whittaker functions.

1. Example: GL_2

First, we consider the smallest example of the Cauchy identity, namely, for $GL_2(\mathbb{C})$.

The identity^[2]

$$\sum_{n=0}^{\infty} z^n \cdot \left(\frac{a^{n+1} - a^{-(n+1)}}{a - a^{-1}} \right) \left(\frac{b^{n+1} - b^{-(n+1)}}{b - b^{-1}} \right) = \frac{1 - z^2}{(1 - zab)(1 - zab^{-1})(1 - za^{-1}b)(1 - za^{-1}b^{-1})}$$

is easily proven by elementary algebra and summing geometric series. Slightly more generally, absorbing z into the other indeterminates,

$$\sum_{n=0}^{\infty} \left(\frac{a^{n+1} - c^{n+1}}{a - c} \right) \left(\frac{b^{n+1} - d^{n+1}}{b - d} \right) = \frac{1 - abcd}{(1 - ab)(1 - ad)(1 - cb)(1 - cd)}$$

for a, b, c, d small, or viewing everything as taking place in a formal power series ring. The above is the simplest instance of *Cauchy's identity*, involving the smallest *Schur polynomials*

$$\frac{a^{n+1} - c^{n+1}}{a - c} = a^n + a^{n-1}c + a^{n-2}c^2 + \dots + a^2c^{n-2} + ac^{n-1} + c^n$$

In fact, the above identities have meaning: we claim that the rearrangement

$$\sum_{m,n \geq 0} (ac)^m \left(\frac{a^{n+1} - c^{n+1}}{a - c} \right) (bd)^m \left(\frac{b^{n+1} - d^{n+1}}{b - d} \right) = \frac{1}{(1 - ab)(1 - ad)(1 - cb)(1 - cd)}$$

[1] The Cauchy identity is often treated as a combinatorial result, in contexts where Schur functions are defined by *formulas*, rather than acknowledged to be *traces* of highest-weight *representations*.

[2] This identity also arises in the non-archimedean local factors of the GL_2 Rankin-Selberg integrals: the two factors in the infinite sum are values of spherical p -adic Whittaker functions.

computes the *trace* of the representation of $GL_2(\mathbb{C}) \times GL_2(\mathbb{C})$ on the space $\mathbb{C}[E]$ of holomorphic polynomials on the set E of two-by-two complex matrices, in two ways. First, let $g \times h \in GL_2(\mathbb{C}) \times GL_2(\mathbb{C})$ act on a polynomial f on E by

$$(g \times h)f(x) = f(g^\top xh)$$

Using transpose rather than inverse in the left action is a convenience: the function $g \rightarrow (g \times h)f(x)$ is thereby a *polynomial* function in g . On the left-hand side of the identity above,

$$\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \rightarrow (ac)^m \left(\frac{a^{n+1} - c^{n+1}}{a - c} \right)$$

is the *character* (that is, *trace*) $\chi_{m,n}$ of the representation $\pi_{m,n} = \det^m \otimes \text{Sym}^n(\text{std})$ of GL_2 , where $\text{Sym}^n(\text{std})$ is the n^{th} symmetric power of the standard representation *std*. The representation is *polynomial* if and only if $m \geq 0$. The character of the representation $\pi_{m,n} \otimes \pi_{m,n}$ of $GL_2(\mathbb{C}) \times GL_2(\mathbb{C})$ is the product:

$$\text{trace}(\pi_{m,n}(g) \otimes \pi_{m,n}(h)) = \chi_{m,n}(g) \cdot \chi_{m,n}(h)$$

Thus, the left-hand side of the identity is

$$\text{tr} \left(\bigoplus_{m,n \geq 0} \pi_{m,n} \otimes \pi_{m,n} \right) = \sum_{m,n \geq 0} \text{tr}(\pi_{m,n} \otimes \pi_{m,n}) = \sum_{m,n \geq 0} \chi_{m,n} \cdot \chi_{m,n}$$

Thus, we must verify (below) that

$$\mathbb{C}[E] \approx \bigoplus_{m,n \geq 0} \pi_{m,n} \otimes \pi_{m,n} \quad (\text{as } GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \text{ representations})$$

On the other hand, we can look at the subgroups

$$M = \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \right\} \subset GL_2(\mathbb{C}) \quad \text{and} \quad M \times M \subset GL_2(\mathbb{C}) \times GL_2(\mathbb{C})$$

and decompose $\mathbb{C}[E]$ as a representation of $M \times M$. Monomials

$$f_{p,q,r,s} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = x^p y^q z^r w^s$$

are obvious *weight vectors*, that is, vector in one-dimensional representations for the action of $M \times M$:

$$\left(\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \times \begin{pmatrix} b & 0 \\ 0 & d \end{pmatrix} \right) f_{p,q,r,s} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = a^{p+q} b^{p+r} c^{r+s} d^{q+s} \cdot f_{p,q,r,s} \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

Thus, it is easy to decompose $\mathbb{C}[E]$ into one-dimensional irreducibles over $M \times M$, and the trace is

$$\begin{aligned} \sum_{p,q,r,s \geq 0} a^{p+q} b^{p+r} c^{r+s} d^{q+s} &= \sum_p (ab)^p \cdot \sum_q (ad)^q \cdot \sum_r (bc)^r \cdot \sum_s (cd)^s \\ &= \frac{1}{(1-ab)(1-ad)(1-bc)(1-cd)} = \frac{1}{\det \left(1_4 - \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \otimes \begin{pmatrix} b & 0 \\ 0 & d \end{pmatrix} \right)} \quad (\text{for small } a, b, c, d) \end{aligned}$$

Thus, if we verify the claim about decomposition of $\mathbb{C}[E]$ as an $GL_2(\mathbb{C}) \times GL_2(\mathbb{C})$ representation, we will have

$$\sum_{m,n \geq 0} \chi_{m,n}(g) \otimes \chi_{m,n}(h) = \frac{1}{\det(1_4 - g \otimes h)} \quad (\text{for } g, h \text{ diagonal, small})$$

Both sides are conjugation-invariant, so we have equality for small *diagonalizable* matrices. The density of diagonalizable matrices in $GL_2(\mathbb{C})$ and continuity imply that the identity holds for all small elements of $GL_2(\mathbb{C}) \times GL_2(\mathbb{C})$.

It is clear that only non-negative weights can appear in a decomposition of the space of polynomials $\mathbb{C}[E]$ as $M \times M$ representation space, but it is less clear that *all* holomorphic irreducibles of $GL_2(\mathbb{C}) \times GL_2(\mathbb{C})$ with exclusively non-negative weights *do* appear. This will be proven after some preparation.

2. $GL_n(\mathbb{C})$ and $U(n)$

The group $K = U(n)$ is a maximal compact subgroup of $G = GL_n(\mathbb{C})$. However, for present purposes, the critical relationship is that any holomorphic function on G vanishing identically when restricted to K is already identically 0 on G . This will imply that the restriction from G to K of an irreducible *holomorphic* representation of G remains irreducible. Further, we will see that *every* irreducible of K occurs as the restriction from G of an irreducible holomorphic representation.

[2.1] Realforms The conclusions about restriction of holomorphic functions will follow from the fact that the Lie algebra $\mathfrak{k} = \mathfrak{u}(n)$ of K is a *realform*^[3] of the Lie algebra $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. To see that \mathfrak{k} is a realform of \mathfrak{g} , observe that \mathfrak{k} consists of skew-hermitian matrices. The \mathbb{R} -dimension of \mathfrak{k} is equal to the \mathbb{C} -dimension of \mathfrak{g} , and any complex n -by- n matrix z can be written as a \mathbb{C} -linear combination of elements of \mathfrak{k} , from the familiar identity

$$z = \frac{z + z^*}{2} + i \cdot \frac{z - z^*}{2i} = i \cdot \frac{z + z^*}{2i} + \frac{z - z^*}{2}$$

where $z \rightarrow z^*$ is conjugate-transpose. This proves the realform property.

[2.2] Restrictions of holomorphic functions We claim that a holomorphic function on G vanishes identically on K if and only if it is identically zero on G . Subordinate to this, we claim that a holomorphic function on \mathfrak{g} vanishes identically on \mathfrak{k} if and only if it is identically zero on \mathfrak{g} .

Since G is *connected*, a holomorphic function on it is completely determined by its power series expansion at the identity $e \in G$. The exponential map $\mathfrak{g} \rightarrow G$ is holomorphic, and is a local isomorphism near $0 \in \mathfrak{g}$ and $e \in G$. Thus, a holomorphic function f near $e \in G$ gives a holomorphic function $F(x) = f(e^x)$ for x near 0 in \mathfrak{g} , and vice-versa. That is, f is identically 0 if and only if F is identically 0. This reduces the first claim about restriction from G to K to the second claim about restriction from \mathfrak{g} to \mathfrak{k} .

The coefficients of the complex power series expansion at 0 of a holomorphic function on a complex vector space V are essentially complex derivatives of the function. All these complex derivatives can be computed along a realform V_o of V , exactly because the realform spans the complex vectorspace. Thus, if the restriction of a holomorphic function to V_o is identically 0, all its real derivatives along V_o are 0, so its complex derivatives at 0 are 0. Thus, its complex power series expansion at 0 is identically 0. Since V is connected, the function is identically 0.

[2.3] Irreducibility criterion via coefficient functions We say that a representation π of $G = GL_n(\mathbb{C})$ is **holomorphic** when $g \rightarrow g \cdot v$ is a holomorphic π -valued function on G , for every $v \in \pi$. For finite-dimensional π this property is immediately equivalent^[4] to holomorphy of the \mathbb{C} -valued coefficient functions

$$c_{v,\mu}^\pi(g) = \langle g \cdot v, \mu \rangle = \mu(gv) \quad (\text{for } v \in \pi \text{ and } \mu \in \pi^*)$$

[3] To assert that a real vector subspace V of a complex vector space W is a *real form* of W is to assert that $\dim_{\mathbb{R}} V = \dim_{\mathbb{C}} W$ and that the *complex* span of V is all of W . Equivalently, the natural complex-linear map $V \otimes_{\mathbb{R}} \mathbb{C} \rightarrow W$ is an isomorphism.

[4] Holomorphy of coefficient functions implies holomorphy of the representation for many infinite-dimensional representations, as well, simply because *weak* holomorphy implies (strong) holomorphy for a broad class of topological

By the above discussion, for $v \neq 0$ and $\mu \neq 0$, the non-zero coefficient function $c_{v,\mu}^\pi$ on G restricts to a non-zero function on K . This restriction is obviously the v, μ coefficient function of the restriction $\text{Res}_K^G \pi$ of π to K .

We claim that a representation τ is irreducible if and only if no one of its coefficients $c_{v,\mu}^\tau$ vanishes identically, of course for $v \neq 0$ and $\mu \neq 0$.

Indeed, for such v, μ , if $c_{v,\mu}^\tau$ were identically 0, then the span of images gv would be inside the kernel of μ , a *proper* closed subspace of τ , contradicting the irreducibility of τ . Conversely, if τ had a proper closed subspace, there would be a continuous linear functional [5] μ whose kernel contains that subspace, so $c_{v,\mu}^\tau$ would be identically 0 for v in that subspace. This proves the irreducibility criterion.

[2.4] **Restriction of holomorphic representations from G to K** Thus, since non-zero coefficient functions of irreducible holomorphic representations π of G restrict to non-zero functions on K , the restriction $\text{Res}_K^G \pi$ of π to K remains irreducible.

We will show just below that *every* irreducible of K occurs as the restriction from G of a holomorphic representation. [6]

[2.5] **Highest weights** The irreducibles of $K = U(n)$ are parametrized by *highest weights* λ , in the following sense. Specifically, let \mathfrak{a} be the standard Cartan subalgebra of diagonal elements of the complexification $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ of \mathfrak{k} , and \mathfrak{n} the strictly upper-triangular elements of \mathfrak{g} . The irreducible complex representations of K are *finite-dimensional*, so \mathfrak{k} and \mathfrak{g} act on them [7] by taking derivatives: for $\theta \in \mathfrak{k}$, $v \in \tau$,

$$\theta \cdot v = \left. \frac{\partial}{\partial t} \right|_{t=0} e^{t\theta} \cdot v$$

The action of \mathfrak{g} is the \mathbb{C} -linear extension.

In an irreducible τ of K , the space $\tau^\mathfrak{n}$ of vectors annihilated by \mathfrak{n} is *one-dimensional*. Necessarily \mathfrak{a} acts on $\tau^\mathfrak{n}$ by a character $\lambda = \lambda_\tau : \mathfrak{a} \rightarrow \mathbb{C}$. The subspace $\tau^\mathfrak{n}$ is the space of *highest weight vectors* in τ , and λ is the *highest weight*.

The map $\tau \rightarrow \lambda$ is *injective*: the highest weight of an irreducible determines the isomorphism class of the irreducible.

Not every $\lambda \in \mathfrak{a}^*$ appears as a highest weight of an irreducible of K . There are two conditions necessary (and sufficient) for λ to appear as a highest weight. Any $\lambda \in \mathfrak{a}^*$ is of the form

$$\lambda \begin{pmatrix} \alpha_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \alpha_n \end{pmatrix} = \lambda_1 \alpha_1 + \dots + \lambda_n \alpha_n \quad (\text{with } \lambda_i \in \mathbb{C})$$

vector spaces. Namely, weak holomorphy implies holomorphy when for functions taking values in a locally convex and quasi-complete topological vector space. The argument uses a variant of the Banach-Steinhaus theorem, and Gelfand-Pettis integrals.

[5] For finite-dimensional π this is easy, and the same conclusion holds for locally convex representation spaces, by Hahn-Banach.

[6] Of course, the holomorphic representations of G that restrict to irreducibles of K are finite-dimensional and irreducible, as representations of G , since they are finite-dimensional and irreducible as representations of K .

[7] In infinite-dimensional representations of Lie groups, the collection of vectors upon which the Lie algebra can act is *dense*, but typically *not* the whole representation space. For finite-dimensional spaces this issue does not arise.

Then λ appears as a highest weight of an irreducible of K if and only if

$$\begin{cases} \text{all } \lambda_j \in \mathbb{Z} & \text{(Integrality)} \\ \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n & \text{(Dominance)} \end{cases}$$

[2.6] **Surjectivity of restriction from G to K** Now we can show that *every* irreducible τ of K is the restriction of a holomorphic representation π of G , and uniquely so. Specifically, given a *dominant, integral* highest weight λ , we construct a finite-dimensional holomorphic representation π_λ of G which restricts to the irreducible τ_λ of K having highest weight λ .

Since λ is *integral*, we can tensor τ_λ with an integer power of *det* to assume that $\lambda_n \geq 0$.

Let G act on holomorphic polynomials f on n -by- n complex matrices x by right translation:

$$(g \cdot f)(x) = f(xg)$$

Letting $x^{(j)}$ be the upper left j -by- j minor of x , let

$$f_\lambda(x) = (\det x^{(1)})^{\lambda_1 - \lambda_2} (\det x^{(2)})^{\lambda_2 - \lambda_3} (\det x^{(3)})^{\lambda_3 - \lambda_4} \dots (\det x^{(n-1)})^{\lambda_{n-1} - \lambda_n} (\det x^{(n)})^{\lambda_n}$$

Note that the integrality of λ , and $\lambda_n \geq 0$, makes f_λ a genuine polynomial. By design, for p in the standard minimal parabolic P of upper-triangular matrices,

$$(p \cdot f_\lambda)(x) = f_\lambda(xp) = p_1^{\lambda_1} p_2^{\lambda_2} \dots p_n^{\lambda_n} \cdot f_\lambda(x)$$

where p_j is the j^{th} diagonal entry of p . Thus, \mathfrak{n} annihilates f_λ , and \mathfrak{a} acts on f_λ by the weight $\lambda = (\lambda_1, \dots, \lambda_n)$. Take π_λ to be the G -subrepresentation of the space of holomorphic polynomials generated by f_λ .

Thus, by construction, $\text{Res}_K^G \pi_\lambda$ contains a copy of the irreducible τ_λ of K with highest weight λ , with highest weight vector f_λ . We claim that the restriction is *exactly* τ_λ .

Since the whole space of holomorphic polynomials in $x \in E$ of a given total degree is finite-dimensional, certainly π_λ is finite-dimensional. Use the Iwasawa decomposition

$$G = K \cdot A^+ \cdot N$$

where A^+ is positive real diagonal elements of G and N is upper-triangular unipotent elements. The exponential map from the Lie algebra of A^+ to A^+ is obviously surjective. It is less obvious, but an easy induction, that the exponential map from the Lie algebra \mathfrak{n} to N surjects. Thus, [8] the G -orbit of v_o is equal to the K -orbit of v_o :

$$G \cdot v_o = K \cdot A^+ \cdot N \cdot v_o = K \cdot \mathbb{C}v_o$$

By *complete reducibility* for finite-dimensional representations of compact groups, any K -representation generated by a highest weight vector is irreducible. [9] Thus, the holomorphic G -representation π_λ so-

[8] For $x \in \mathfrak{g}$ and v a smooth vector in a representation space π , the smooth π -valued function $F(t) = e^{tx} \cdot v$ satisfies the differential equation $F' = x \cdot F$, with initial condition $F(0) = v$. The Mean Value Theorem proves uniqueness of the solution. Thus, assuming existence of the *group* representation π , the *algebra* action of $x \in \mathfrak{g}$ determines the *group* action of e^{tx} completely.

[9] Let σ be a finite-dimensional K -representation generated by a highest weight vector v_o . By complete reducibility, σ is a direct sum of K -irreducibles σ_j . Each σ_j has a unique (up to scalars) highest weight vector w_j with weight λ_j . Thus, v_o is a linear combination of those w_j 's such that $\lambda_j = \lambda$. That is, since v_o generates σ , in fact, *all* w_j 's have $\lambda_j = \lambda$, and the σ_j 's are mutually isomorphic. Scaling the w_j as necessary, we can write $v_o = \sum_j w_j$. Let $\varphi_j : \sigma_1 \rightarrow \sigma_j$ be the unique K -isomorphism taking w_1 to w_j . The K -orbit of v_o consists of elements

$$k \cdot v_o = \sum_j k \cdot w_j = \sum_j k \cdot \varphi_j(w_1) = \left(\sum_j \varphi_j \right) (k \cdot w_1)$$

That is, v_o generates a K -homomorphic image of σ_1 , so is irreducible. Thus, in fact, there is just one summand σ_j .

constructed restricts to τ_λ on K .

That is, given the irreducible τ_λ of K with (dominant integral) highest weight λ , with $\lambda_1 \geq \dots \geq \lambda_n \geq 0$, the holomorphic polynomial representation π_λ constructed above restricts to τ_λ .

Dropping the assumption $\lambda_n \geq 0$, a similar conclusion follows by tensoring with integer powers of determinant, but the *polynomial* character of π_λ seems to be lost when $\lambda_n < 0$, whether or not λ is dominant integral.

That is, whether or not $\lambda_n \geq 0$, every irreducible τ_λ of K is the restriction of a holomorphic irreducible π_λ of G . If $\lambda_n \geq 0$, then π_λ occurs as a subrepresentation of holomorphic polynomials $\mathbb{C}[E]$ on n -by- n complex matrices E .

[2.7] **Extendability of representations of G** Since $G = GL_n(\mathbb{C})$ is a dense open subset of the complex vector space E of n -by- n complex matrices, for holomorphic (or other) representations π of G it is reasonable to ask whether π *extends* continuously to a holomorphic $\text{End}_{\mathbb{C}}(\pi)$ -valued function on E .

For *finite-dimensional* holomorphic representations, extendability of π is equivalent to extendability of all coefficient functions of π .

In fact, for finite-dimensional holomorphic representations, the highest-weight classification above shows that for any finite-dimensional holomorphic π , there is an integer ℓ such that $\det^\ell \otimes \pi$ is *polynomial*, so certainly extends to a holomorphic function on E .

Specifically, the finite-dimensional holomorphic representations π_λ with dominant integral highest weights $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_n \geq 0$ are polynomial, and extend to E .

We claim that π_λ does *not* extend when $\lambda_n < 0$. Recall that the rational function

$$f_\lambda(x) = (\det x^{(1)})^{\lambda_1 - \lambda_2} (\det x^{(2)})^{\lambda_2 - \lambda_3} (\det x^{(3)})^{\lambda_3 - \lambda_4} \dots (\det x^{(n-1)})^{\lambda_{n-1} - \lambda_n} (\det x^{(n)})^{\lambda_n}$$

generates π_λ under the right action of G , regardless of the sign of λ_n . Specifically, for upper-triangular p ,

$$f_\lambda(xp) = p_1^{\lambda_1} p_2^{\lambda_2} \dots p_n^{\lambda_n} \cdot f_\lambda(x)$$

where p_i is the i^{th} diagonal entry of p . For $\lambda_n < 0$, this function cannot extend to E , because for fixed non-zero p_1, \dots, p_{n-1} it blows up as p_n goes to 0. This proves the claim.

Thus, further, if a single (non-zero) coefficient function of π^λ extends, then *all* coefficient functions extend. Indeed, a coefficient function $c_{\nu}^{\pi}(g)$ that does *not* extend is not polynomial, so $\lambda_n < 0$, and π_λ does not extend.

3. *Decomposing holomorphic polynomials over $GL_n \times GL_n$*

The Cauchy identity is obtained by taking the trace of a representation in two ways. The analogue of the Cauchy identity for bigger groups $GL_n(\mathbb{C})$ cannot use explicit elementary expressions for characters of finite-dimensional irreducibles, as did the $GL_2(\mathbb{C})$ example.

[3.1] **Schur polynomials** Without trying to write an explicit expression ^[10] for it, take the λ^{th} **Schur polynomial** $s_\lambda(t_1, \dots, t_n)$ to be the *character*

$$s_\lambda(t_1, \dots, t_n) = \text{trace} \left(\pi_\lambda \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \right)$$

[10] The Weyl character formula gives a *sort* of explicit expression for the Schur polynomials, but that expression is not useful in proving the Cauchy identity here.

of the holomorphic irreducible π_λ of $GL_n(\mathbb{C})$ with highest weight^[11] λ , evaluated on diagonal matrices with entries t_1, \dots, t_n .

[3.2] The polynomial representation Let $G \times G$ act on holomorphic polynomials $\mathbb{C}[E]$ on the collection E of complex n -by- n matrices, by

$$(g \times h)f(x) = f(g^\top \cdot x \cdot h)$$

Note that the left action has *transpose* in it, rather than *inverse*. We do this to have the special polynomial

$$f_\lambda(x) = (\det x^{(1)})^{\lambda_1 - \lambda_2} (\det x^{(2)})^{\lambda_2 - \lambda_3} (\det x^{(3)})^{\lambda_3 - \lambda_2} \dots (\det x^{(n-1)})^{\lambda_{n-1} - \lambda_n} (\det x^{(n)})^{\lambda_n}$$

be acted upon on the *left* by P by

$$f_\lambda(p^\top x) = p_1^{\lambda_1} p_2^{\lambda_2} \dots p_n^{\lambda_n} \cdot f_\lambda(x)$$

in addition to the corresponding action of P on the right noted above. Thus, the representation of $G \times G$ generated by f_λ restricts to the irreducible representation of $K \times K$ with highest weight $\lambda \oplus \lambda$.

In particular, the representation $\pi_\lambda \otimes \pi_\lambda$ occurs in $\mathbb{C}[E]$ at least once. This is a *lower* bound on the decomposition of $\mathbb{C}[E]$ into irreducibles.

To obtain an *upper* bound on $\mathbb{C}[E]$, observe that the restriction of functions from G to K is *injective* on *holomorphic* functions. Thus, under restriction of functions, $\mathbb{C}[E]$ *injects* to smooth functions on K . Further, since $\mathbb{C}[E]$ consists of *polynomials*, every vector in it is $G \times G$ -*finite*^[12] and, hence, $K \times K$ -finite. In the biregular representation of $K \times K$ on functions on K , the collection of $K \times K$ -finite vectors is exactly the algebraic direct sum

$$\sum_{\lambda} \tau_\lambda \otimes \tau_\lambda^\vee \quad (\text{summed over dominant integral highest weights})$$

where τ_λ is the irreducible of K with highest weight λ .

Since the left action of G on $\mathbb{C}[E]$ is not $(g \cdot f)(x) = f(g^{-1}x)$, but $(g \cdot f)(x) = f(g^\top x)$, we might want to reconcile this with the convention for the biregular representation of $K \times K$. We claim that for an irreducible τ of K , replacing $k \cdot v = \tau(g)(v)$ by $\tau((k^\top)^{-1})(v)$ yields the *contragredient* τ^\vee to τ . To see this, use the classification by highest weights. Indeed, taking transpose converts a *highest* weight to a *lowest* weight, and vice-versa. The theory of highest weights shows that *finite-dimensional* highest-weight representations have lowest weight equal to the negative of the highest weight of the *contragredient*. Thus, taking inverse (after taking transpose) gives a representation of K with the same highest weight as the contragredient τ^\vee , hence, isomorphic to τ^\vee .

Thus, with inverse replaced by transpose for the left action, the restriction of the representation $\mathbb{C}[E]$ of $G \times G$ to a representation of $K \times K$ necessarily gives a subrepresentation of

$$\sum_{\lambda} \tau_\lambda \otimes \tau_\lambda$$

[11] This use of *highest weight* is a potentially confusing slight abuse of language, since the *true* weights for $GL_n(\mathbb{C})$ would necessarily include *all* characters of the Cartan subgroup (diagonal matrices), not merely holomorphic ones. But restriction to $U(n)$ sends irreducibles to irreducibles, and it is *literally* correct to say that the restriction of π_λ to $U(n)$ has highest weight λ .

[12] As usual, a vector v in a representation π of a group G is G -*finite* when the span of the images $g \cdot v$ is finite-dimensional.

Since π_λ restricts to τ_λ , this implies that $\pi_\lambda \otimes \pi_\lambda$ occurs *at most* once in $\mathbb{C}[E]$, giving the desired upper bound on multiplicities.

Finally, since the representation of $G \times G$ on $\mathbb{C}[E]$ is polynomial, it certainly *extends* to $E \times E$, as do coefficient functions of its irreducible subrepresentations. From the earlier discussion of extendability, the only representations $\pi_\lambda \otimes \pi_\lambda$ appearing must have $\lambda_n \geq 0$. We have already observed that for $\lambda_n \geq 0$, the special polynomial f_λ certainly is in $\mathbb{C}[E]$, and generates a copy of $\pi_\lambda \otimes \pi_\lambda$. Thus,

$$\mathbb{C}[E] \approx \sum_{\lambda : \lambda_n \geq 0} \pi_\lambda \otimes \pi_\lambda$$

[3.3] Taking trace: the Cauchy identity Though $\mathbb{C}[E]$ is an infinite-dimensional representation of $G \times G$, it is an ascending union, or colimit, of finite-dimensional representations, so has a *trace*.

On one hand, expressing the trace in terms of the irreducible summands $\pi_\lambda \otimes \pi_\lambda$ gives

$$\text{trace } \mathbb{C}[E] = \sum_{\lambda : \lambda_n \geq 0} \text{tr}(\pi_\lambda \otimes \pi_\lambda) = \sum_{\lambda : \lambda_n \geq 0} \text{tr} \pi_\lambda \cdot \text{tr} \pi_\lambda = \sum_{\lambda : \lambda_n \geq 0} s_\lambda \cdot s_\lambda$$

On the other hand, the *monomials* in $\mathbb{C}[E]$ are weight vectors for the subgroup $M \times M$ of diagonal matrices in $G \times G$. Thus, evaluating the trace on diagonal elements

$$a \times b = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \times \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{pmatrix}$$

$$(\text{trace } \mathbb{C}[E])(a \times b) = \sum_{n_{ij} \geq 0} \prod_{i,j=1}^n (a_i b_j)^{n_{ij}} = \prod_{i,j=1}^n \frac{1}{1 - a_i b_j} = \frac{1}{\det(1_{n^2} - a \otimes b)}$$

Thus, we have Cauchy's identity for diagonal elements:

$$\sum_{\lambda : \lambda_n \geq 0} s_\lambda(a) \cdot s_\lambda(b) = \frac{1}{\det(1_{n^2} - a \otimes b)}$$

either as a formal power series identity or for convergent power series with a, b small. Both sides are conjugation-invariant, so the identity holds for *diagonalizable* elements of $G \times G$. These are *dense*, so the identity holds throughout:

$$\sum_{\lambda : \lambda_n \geq 0} s_\lambda(g) \cdot s_\lambda(h) = (\text{tr } \mathbb{C}[E])(g \times h) = \frac{1}{\det(1_{n^2} - g \otimes h)} \quad (\text{for } g, h \in G)$$

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