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Analytic continuation, functional equation: examples

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1. $L(s,\chi)$ for *even* Dirichlet characters
2. $L(s,\chi)$ for *odd* Dirichlet characters
3. Dedekind zeta function $\zeta_{\mathfrak{o}}(s)$ for Gaussian integers $\mathfrak{o}$
4. Grossencharacter L-functions L-functions for Gaussian integers

We try to imitate the argument used by Riemann for proving the *analytic continuation* of $\zeta(s)$ and its *functional equation*

$$\pi^{-s/2}\,\Gamma(\tfrac{s}{2})\,\zeta(s) \;=\; \pi^{-(1-s)/2}\,\Gamma(\tfrac{1-s}{2})\,\zeta(1-s)$$

from the *integral representation*

$$\pi^{-s/2}\,\Gamma(\tfrac{s}{2})\,\zeta(s) \;=\; \int_0^\infty y^{-s/2}\,\frac{\theta(iy)-1}{2}\,\frac{dy}{y}$$

in terms of the basic *theta series*[1]

$$\theta(iy) \;=\; \sum_{n\in\mathbb{Z}} e^{-\pi n^2 y}$$

whose functional equation

$$\theta(iy) \;=\; \frac{1}{\sqrt{y}}\cdot\theta(\frac{i}{y})$$

is proven by *Poisson summation.*

The discussion of L-functions $L(s,\chi)$ for Dirichlet characters (over $\mathbb{Q}$) bifurcates into two families, depending upon the *parity* of χ, that is, whether $\chi(-1)=+1$ or $\chi(-1)=-1$.

Analogous discussion of the zeta function of the Gaussian integers extends this discussion in a new direction. Treatment of the *grossencharacter* L-functions [2] for the Gaussian integers have no counterpart among Dirichlet L-functions for $\mathbb{Z}$.

L-functions of *ideal class group* characters deserve parallel treatment, but would require more background, concerning rings of algebraic integers that fail to be principal ideal domains, and other background. This background is important, but is not the immediate point, so we defer this part of the discussion.

These examples are simple, but the book-keeping quickly becomes fragile. This should motivate receptiveness to the more abstract, but very clear, Iwasawa-Tate viewpoint, which we consider soon.

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1. *Dirichlet L-functions $L(s,\chi)$ for even Dirichlet characters*

Let χ be a *non-trivial* Dirichlet character mod $N>1$, with L-function

$$L(s,\chi) \;=\; \sum_{n\geq 1}\frac{\chi(n)}{n^s}$$

[1] The reason that the argument is given as iy rather than y is that iy can be replaced by any z in the complex upper half-plane $\mathfrak{H}$. This aspect is important later.

[2] The *grossencharacters* and their L-functions are also called *Hecke characters* and L-functions, as Hecke first studied them, about 1920.

By imitation of the corresponding discussion for $\zeta(s)$, we expect to define a *theta series* θ_χ with a *functional equation* provable via Poisson summation, to exhibit an *integral representation* of $L(s,\chi)$ in terms of θ_χ, and to use this integral representation to prove the analytic continuation and see the functional equation in the symmetry of the rewritten integral representation.

We will find that the most obvious imitative approach succeeds only for *even* χ. Further, to have a symmetrical form of the functional equation, we must be able to determine the absolute value of the Gauss sum

$$\langle \chi, \psi \rangle \;=\; \sum_{b \bmod N} \chi(b)\,\overline{\psi}(b) \qquad\qquad (\text{with } \psi(b) = e^{2\pi i b/N})$$

This requires that χ be *primitive* mod N, that is, its *conductor* must be exactly N, not a proper divisor. That is, the non-zero parts of χ must not be well-defined modulo N' for any proper divisor of N.

[1.1] The theta series, parity of χ

The obvious imitation of the theta series for ζ suggests defining

$$\theta_\chi(iy) \;=\; \sum_{n\in\mathbb{Z}} \chi(n)\, e^{-\pi n^2 y}$$

Non-trivial χ is extended by 0 to integers having a common factor with N, so $\chi(0) = 0$. By design,

$$\frac{\theta_\chi(iy)}{2} \;=\; \sum_{n\geq 1} \frac{\chi(n)+\chi(-n)}{2}\, e^{-\pi n^2 y} \;=\; \begin{cases} 0 & (\text{for } \chi \text{ odd}) \\ \sum_{n\geq 1} \chi(n)\, e^{-\pi n^2 y} & (\text{for } \chi \text{ even}) \end{cases}$$

That is, this version of θ_χ cannot be correct for *odd* χ, since it vanishes identically.

[1.2] The integral representation

The integral representation is easy: for $\mathrm{Re}(s) > 1$,

$$\int_0^\infty y^{\frac{s}{2}}\, \frac{\theta_\chi(iy)}{2}\, \frac{dy}{y} \;=\; \sum_{n\geq 1} \chi(n) \int_0^\infty y^{\frac{s}{2}}\, e^{-\pi n^2 y}\, \frac{dy}{y}$$

$$=\; \sum_{n\geq 1} \frac{\chi(n)}{\pi^{s/2}\, n^s} \int_0^\infty y^{\frac{s}{2}}\, e^{-y}\, \frac{dy}{y} \;=\; \pi^{-s/2}\, \Gamma(\tfrac{s}{2})\, L(s,\chi)$$

by replacing y by $y/\pi n^2$. This is the desired integral representation.

[1.2.1] Remark: The measure dy/y is dilation-invariant, and also invariant under any change of variables $y \to y/C$ for $C > 0$. Since these are the changes of variables the computation will require, it is vastly better to keep the y in the denominator of the measure, rather than absorb it into the $y^{s/2}$.

[1.3] Functional equation of θ_χ

We anticipate that θ_χ has a functional equation similar to that of $\theta(iy) = \sum_n e^{-\pi n^2 y}$, proven by application of the Poisson summation formula

$$\sum_{n\in\mathbb{Z}} f(n) \;=\; \sum_{n\in\mathbb{Z}} \widehat{f}(n) \qquad\qquad (\text{for nice functions } f)$$

The minor but meaningful obstacle is that in this classical guise θ_χ is *not* simply a sum of a nice function over a lattice, so Poisson summation is not immediately applicable. However, since $\chi(n)$ depends only upon n mod N, we can break the sum expressing θ_χ into a finite sum of sums-over-lattices, and then apply Poisson summation.

Specifically,

$$\theta_\chi(iy) \;=\; \sum_n \chi(n)\, e^{-\pi n^2 y} \;=\; \sum_{b \bmod N} \chi(b) \sum_{\ell\in\mathbb{Z}} e^{-\pi(\ell N + b)^2 y}$$

Thus, Poisson summation is to be applied to the function

$$f(x) \;=\; e^{-\pi(xN+b)^2 y}$$

Since this f differs from the Gaussian $e^{-\pi x^2}$ merely by dilations and translation, and the Fourier transform behaves cogently with respect to such, something reasonable will come out, by changing variables:

$$\begin{aligned}
\Big(e^{-\pi(xN+b)^2 y}\Big)\widehat{\ }(\xi) \;&=\; \int_{\mathbb{R}} e^{-\pi(xN+b)^2 y}\, e^{-2\pi\xi x}\, dx \\
&=\; \frac{1}{N}\int_{\mathbb{R}} e^{-\pi(x+b)^2 y}\, e^{-2\pi\xi x/N}\, dx \qquad \text{(replacing } x \text{ by } x/N) \\
&=\; \frac{e^{2\pi i\xi b/N}}{N}\int_{\mathbb{R}} e^{-\pi x^2 y}\, e^{-2\pi\xi x/N}\, dx \qquad \text{(replacing } x \text{ by } x-b) \\
&=\; \frac{e^{2\pi i\xi b/N}}{N\sqrt{y}}\int_{\mathbb{R}} e^{-\pi x^2}\, e^{-2\pi\xi x/N\sqrt{y}}\, dx \qquad \text{(replacing } x \text{ by } /\sqrt{y}) \\
&=\; \frac{e^{2\pi i\xi b/N}}{N\sqrt{y}} e^{-\pi\xi^2/N^2 y} \qquad \text{(taking Fourier transform)}
\end{aligned}$$

Thus, by Poisson summation,

$$\theta_\chi(iy) \;=\; \sum_{b \bmod N} \chi(b) \sum_{\ell\in\mathbb{Z}} e^{-\pi(\ell N+b)^2 y} \;=\; \sum_{b \bmod N} \chi(b) \sum_{\ell\in\mathbb{Z}} \frac{e^{2\pi i\ell b/N}}{N\sqrt{y}} e^{-\pi\ell^2/N^2 y}$$

$$=\; \frac{1}{N\sqrt{y}} \sum_{\ell\in\mathbb{Z}} e^{-\pi\ell^2/N^2 y} \sum_{b \bmod N} \chi(b) e^{2\pi i\ell b/N}$$

The obvious thing would be to replace b by $b\ell^{-1}$ mod N, but the sum over ℓ does not obviously exclude ℓ with $\gcd(\ell, N) > 1$. This is the first point where the *primitivity* of χ mod N enters, to make the inner sum vanish for $\gcd(\ell, N) > 1$:

[1.3.1] Claim:

$$\sum_{b \bmod N} \chi(b)\, e^{2\pi i\ell b/N} \;=\; 0 \qquad \text{(for } \textit{primitive } \chi \bmod N\text{, for } \ell \textit{ not } \text{invertible mod } N)$$

Proof: Let $\gcd(\ell, N) = N/m > 1$. Then, replacing b by $b\cdot(1 + xm$ for any x mod N/m gives

$$\sum_{b \bmod N} \chi(b)\, e^{2\pi i\ell b/N} \;=\;=\; \sum_{b \bmod N/m}\; \sum_{\beta \bmod m} \chi(b(1+xm))\, e^{2\pi i\ell\beta(1+xm)/N}$$

$$=\; \chi(1+xm) \sum_{b \bmod N/m}\; \sum_{\beta \bmod m} \chi(b)\, e^{2\pi i\ell\beta/N}$$

This holds for all x. The primitivity of χ is exactly that $x \to \chi(1 + xm)$ cannot be trivial. Thus, the sum is 0, as claimed. ///

Returning to the functional equation for θ_χ, now we *can* make the change of variables: replace b by $b\ell^{-1}$ mod N:

$$\theta_\chi(iy) \;=\; \frac{1}{N\sqrt{y}} \sum_{\ell\in\mathbb{Z}} \chi^{-1}(\ell)\, e^{-\pi\ell^2/N^2 y} \sum_{b \bmod N} \chi(b) e^{2\pi ib/N}$$

The inner sum has become a Gauss sum $\langle\chi,\psi\rangle$ *not* depending on ℓ,

$$\langle\chi,\psi\rangle \;=\; \sum_{b \bmod N} \chi(b)\,\overline{\psi}(b) \qquad\qquad (\text{with } \psi(b) = e^{-2\pi i b/N})$$

so we have the functional equation:

$$\theta_\chi(iy) \;=\; \frac{\langle\chi,\psi\rangle}{N\sqrt{y}} \cdot \theta_{\chi^{-1}}\Big(\frac{i}{N^2 y}\Big) \qquad\qquad (\text{for even } \chi \text{ primitive mod } N)$$

[1.3.2] Remark: The most striking thing is that the functional equation relates θ_χ to $\theta_{\chi^{-1}}$, rather than to θ_χ itself. This is typical of a larger reality.

[1.3.3] Remark: Note that the *flip* in the argument y is $y \to 1/N^2 y$, not the earlier $y \to 1/y$. The fixed-point of this map is not 1, but $1/N$. This influences the break-up of the integral in proof of the analytic continuation and functional equation, next.

[1.4] Analytic continuation and functional equation

Starting from the integral representation,

$$\pi^{-s/2}\,\Gamma(\tfrac{s}{2})\,L(s,\chi) \;=\; \int_0^\infty y^{\frac{s}{2}}\,\frac{\theta_\chi(iy)}{2}\,\frac{dy}{y} \;=\; \int_{1/N}^\infty y^{\frac{s}{2}}\,\frac{\theta_\chi(iy)}{2}\,\frac{dy}{y} + \int_0^{1/N} y^{\frac{s}{2}}\,\frac{\theta_\chi(iy)}{2}\,\frac{dy}{y}$$

As for $\zeta(s)$, the integral from $1/N$ to ∞ is nicely convergent for all $s \in \mathbb{C}$, and gives an entire function. The goal is to use the functional equation to convert the integral from 0 to $1/N$ to the other sort, and then symmetrize the resulting expression as much as possible. That is, replace y by $1/N^2 y$ in the problemmatical integral from 0 to $1/N$, obtaining

$$\tfrac{1}{2}\int_{1/N}^\infty (1/N^2 y)^{\frac{s}{2}}\,\theta_\chi\Big(\frac{i}{N^2 y}\Big)\,\frac{dy}{y} \;=\; \tfrac{1}{2}\,N^{-s}\int_{1/N}^\infty y^{-\frac{s}{2}}\,\theta_\chi\Big(\frac{i}{N^2 y}\Big)\,\frac{dy}{y}$$

Replacing χ by χ^{-1} in the functional equation above gives

$$\theta_{\chi^{-1}}(iy) \;=\; \frac{\langle\chi^{-1},\psi\rangle}{N\sqrt{y}} \cdot \theta_\chi\Big(\frac{i}{N^2 y}\Big)$$

Rearranging,

$$\theta_\chi\Big(\frac{i}{N^2 y}\Big) \;=\; \frac{N\sqrt{y}}{\langle\chi^{-1},\psi\rangle}\theta_{\chi^{-1}}(iy)$$

Substituting this gives

$$\frac{N^{1-s}}{\langle\chi^{-1},\psi\rangle}\int_{1/N}^\infty y^{\frac{1-s}{2}}\,\frac{\theta_{\chi^{-1}}(iy)}{2}\,\frac{dy}{y}$$

That is,

$$\pi^{-s/2}\,\Gamma(\tfrac{s}{2})\,L(s,\chi) \;=\; \int_{1/N}^\infty y^{\frac{s}{2}}\,\frac{\theta_\chi(iy)}{2}\,\frac{dy}{y} \;+\; \frac{N^{1-s}}{\langle\chi^{-1},\psi\rangle}\int_{1/N}^\infty y^{\frac{1-s}{2}}\,\frac{\theta_{\chi^{-1}}(iy)}{2}\,\frac{dy}{y}$$

This gives the *analytic continuation*, but is not quite symmetrized. Not only is there the inescapable feature that sending s to $1-s$ requires that χ becomes χ^{-1}, but, also, the Gauss sum and the power of the conductor N appear.

Recall that for χ *primitive*,

$$|\langle\chi,\psi\rangle| \;=\; \sqrt{N}$$

Since we do not pretend to know the *argument* of the Gauss sum, only its *size*, let

$$\varepsilon(\chi) \;=\; \frac{\sqrt{N}}{\langle \chi^{-1}, \psi\rangle}$$

Then $|\varepsilon(\chi)| = 1$ and

$$\pi^{-s/2}\,\Gamma(\tfrac{s}{2})\,L(s,\chi) \;=\; \int_{1/N}^\infty \Big(y^{\frac{s}{2}}\,\frac{\theta_\chi(iy)}{2} \;+\; \varepsilon(\chi) N^{\frac{1}{2}-s} y^{\frac{1-s}{2}}\,\frac{\theta_{\chi^{-1}}(iy)}{2}\Big)\,\frac{dy}{y}$$

Ignoring $\varepsilon(\chi)$ for a moment, we *can* symmetrize the powers of the conductor N, by multiplying through by $N^{s/2}$, giving

$$N^{s/2}\,\pi^{-s/2}\,\Gamma(\tfrac{s}{2})\,L(s,\chi) \;=\; \int_{1/N}^\infty \Big((Ny)^{\frac{s}{2}}\,\frac{\theta_\chi(iy)}{2} \;+\; \varepsilon(\chi)(Ny)^{\frac{1-s}{2}}\,\frac{\theta_{\chi^{-1}}(iy)}{2}\Big)\,\frac{dy}{y}$$

The seeming asymmetry in $\varepsilon(\chi)$ is an illusion: since χ is *even*,

$$\overline{\langle \chi^{-1}, \psi\rangle} \;=\; \langle \chi, \overline{\psi}\rangle \;=\; \sum_b \chi(b)\overline{\psi}(-b) \;=\; \chi(-1)\sum_b \chi(b)\overline{\psi}(b) \;=\; \chi(-1)\langle \chi, \psi\rangle \;=\; \langle \chi, \psi\rangle \qquad \text{(for } \chi \text{ even)}$$

That is,

$$\varepsilon(\chi)\cdot\varepsilon(\chi^{-1}) \;=\; \frac{N}{\langle \chi^{-1}, \psi\rangle \cdot \langle \chi, \psi\rangle} \;=\; \frac{N}{\overline{\langle \chi, \psi\rangle} \cdot \langle \chi, \psi\rangle} \;=\; 1$$

Thus, there is the symmetry

$$\varepsilon(\chi)\cdot\varepsilon(\chi^{-1}) \;=\; 1 \qquad \text{(for } \chi \text{ even)}$$

Thus, we have the *functional equation*

$$\begin{aligned} N^{-s/2}\,\pi^{-s/2}\,\Gamma(\tfrac{s}{2})\,L(s,\chi) \;&=\; \int_{1/N}^\infty \Big((Ny)^{\frac{s}{2}}\,\frac{\theta_\chi(iy)}{2} \;+\; \varepsilon(\chi)(Ny)^{\frac{1-s}{2}}\,\frac{\theta_{\chi^{-1}}(iy)}{2}\Big)\,\frac{dy}{y} \\ &=\; \varepsilon(\chi) \int_{1/N}^\infty \Big(\varepsilon(\chi^{-1}(Ny)^{\frac{s}{2}}\,\frac{\theta_\chi(iy)}{2} \;+\; (Ny)^{\frac{1-s}{2}}\,\frac{\theta_{\chi^{-1}}(iy)}{2}\Big)\,\frac{dy}{y} \\ &=\; \varepsilon(\chi)\cdot N^{-(1-s)/2}\,\pi^{-(1-s)/2}\,\Gamma(\tfrac{1-s}{2})\,L(1-s,\chi^{-1}) \qquad \text{(for } \chi \text{ even)} \end{aligned}$$

In summary,

$$N^{s/2}\,\pi^{-s/2}\,\Gamma(\tfrac{s}{2})\,L(s,\chi) \;=\; \varepsilon(\chi)\cdot N^{(1-s)/2}\,\pi^{-(1-s)/2}\,\Gamma(\tfrac{1-s}{2})\,L(1-s,\chi^{-1}) \qquad \text{(for } \chi \text{ even)}$$

where

$$\varepsilon(\chi) \;=\; \frac{\sqrt{N}}{\langle \chi^{-1}, \psi\rangle} \qquad \text{and} \qquad |\varepsilon(\chi)| \;=\; 1, \qquad \varepsilon(\chi)\cdot\varepsilon(\chi^{-1}) \;=\; 1 \qquad (\chi \text{ even})$$

[1.4.1] **Remark**: The appearance of the conductor N is inescapable, as is the appearance of $\varepsilon(\chi)$.

2. *Dirichlet L-functions $L(s,\chi)$ for odd Dirichlet characters*

Now accommodations are made for *odd* χ, *primitive* mod N.

[2.1] The theta series, parity of χ Again, the obvious imitation of the theta series for ζ suggests defining

$$\theta_\chi(iy) \;=\; \sum_{n\in\mathbb{Z}} \chi(n)\, e^{-\pi n^2 y} \qquad\qquad \text{(bad: vanishes identically for odd } \chi!)$$

but for *odd* χ this fails, being identically 0 due to cancellation of the $\pm n$ summands. The Gaussian $e^{-\pi x^2}$ must be modified to be *odd*: the function $xe^{-\pi x^2}$ will do. Thus, let [3]

$$\widetilde{\theta}_\chi(iy) \;=\; \sum_{n\in\mathbb{Z}} \chi(n)\, n\sqrt{y}\, e^{-\pi n^2 y} \qquad\qquad \text{(for odd } \chi)$$

[2.2] The integral representation The integral representation is easy, assuming that we keep track of the little shifts created by the alterations in $\widetilde{\theta}$: for $\mathrm{Re}(s) > 1$,

$$\int_0^\infty y^{\frac{s}{2}}\, \frac{\widetilde{\theta}_\chi(iy)}{2}\, \frac{dy}{y} \;=\; \sum_{n\geq 1} \chi(n) \int_0^\infty y^{\frac{s}{2}}\, n\sqrt{y}\, e^{-\pi n^2 y}\, \frac{dy}{y}$$

$$=\; \sum_{n\geq 1} \frac{\chi(n)\, n}{\pi^{(s+1)/2}\, n^{s+1}} \int_0^\infty y^{\frac{s+1}{2}}\, e^{-y}\, \frac{dy}{y} \;=\; \pi^{-\frac{s+1}{2}}\, \Gamma(\tfrac{s+1}{2})\, L(s,\chi)$$

by replacing y by $y/\pi n^2$. This is the desired integral representation.

[2.2.1] Remark: The shifts in the exponent of π and argument of Γ are inescapable and correct, since the functional equation will turn out to be $s \to 1-s$ *with* these shifts in the gamma factor.

[2.3] Functional equation of $\widetilde{\theta}_\chi$ We anticipate that $\widetilde{\theta}_\chi$ has a functional equation similar to that of $\theta(iy) = \sum_n e^{-\pi n^2 y}$, proven by Poisson summation.

As with *even* χ, the minor obstacle is that $\widetilde{\theta}_\chi$ is *not* simply a sum of a nice function over a lattice, so Poisson summation is not immediately applicable. However, since $\chi(n)$ depends only upon n mod N, we can break the sum expressing $\widetilde{\theta}_\chi$ into a finite sum of sums-over-lattices, and then apply Poisson summation.

Specifically,

$$\widetilde{\theta}_\chi(iy) \;=\; \sum_n \chi(n)\, n\sqrt{y}\, e^{-\pi n^2 y} \;=\; \sum_{b \bmod N} \chi(b) \sum_{\ell\in\mathbb{Z}} (\ell N + b)\sqrt{y}\, e^{-\pi(\ell N+b)^2 y}$$

Thus, Poisson summation is to be applied to

$$f((xN+b)\sqrt{y}) \;=\; (xN+b)\sqrt{y}\, e^{-\pi(xN+b)^2 y} \qquad\qquad (\text{with } f(x) = xe^{-\pi x^2})$$

[3] Given the existing literature, it is important to understand that this is *not* the standard classical-style normalization of the corresponding theta function, whence the modifying tilde. Specifically, the usual classical theta series would omit the $\sqrt{y}$. Of course, such variations can be compensated-for in companion normalizations. The choice here is motivated by somewhat longer-range considerations.

This differs from $f(x)$ merely by dilations and translation, and the Fourier transform behaves cogently with respect to such, so something reasonable will come out. By changing variables:

$$\begin{aligned}
\Big(f((xN+b)\sqrt{y})\Big)\widehat{\ }(\xi) &= \int_{\mathbb{R}} f((xN+b)\sqrt{y})\, e^{-2\pi\xi x}\, dx \\
&= \frac{1}{N}\int_{\mathbb{R}} f((x+b)\sqrt{y})\, e^{-2\pi\xi x/N}\, dx \quad \text{(replacing } x \text{ by } x/N) \\
&= \frac{e^{2\pi i\xi b/N}}{N}\int_{\mathbb{R}} f(x\sqrt{y})\, e^{-2\pi\xi x/N}\, dx \quad \text{(replacing } x \text{ by } x-b) \\
&= \frac{e^{2\pi i\xi b/N}}{N\sqrt{y}}\int_{\mathbb{R}} f(x)\, e^{-2\pi\xi x/N\sqrt{y}}\, dx \quad \text{(replacing } x \text{ by } /\sqrt{y}) \\
&= \frac{e^{2\pi i\xi b/N}}{N\sqrt{y}}\widehat{f}(\xi/N\sqrt{y}) \quad \text{(taking Fourier transform)}
\end{aligned}$$

The function $f(x) = xe^{-\pi x^2}$ is an *eigenfunction*[4] of Fourier transform:

$$\big(xe^{-\pi x^2}\big)\widehat{\ }(\xi) \;=\; -i\cdot \xi e^{-\pi\xi^2}$$

Thus, by Poisson summation,

$$\begin{aligned}
\widetilde{\theta}_\chi(iy) &= \sum_{b \bmod N}\chi(b)\sum_{\ell\in\mathbb{Z}} f((\ell N+b)\sqrt{y}) \;=\; -i\sum_{b \bmod N}\chi(b)\sum_{\ell\in\mathbb{Z}}\frac{e^{2\pi i\ell b/N}}{N\sqrt{y}} f\Big(\frac{\ell}{N\sqrt{y}}\Big) \\
&= \frac{-i}{N\sqrt{y}}\sum_{\ell\in\mathbb{Z}} f\Big(\frac{\ell}{N\sqrt{y}}\Big)\sum_{b \bmod N}\chi(b)e^{2\pi i\ell b/N}
\end{aligned}$$

As already proven in the treatment of *even* χ, for *primitive* χ the inner sum *vanishes* for $\gcd(\ell, N) > 1$.

Returning to the functional equation for $\widetilde{\theta}_\chi$, now we *can* make the change of variables: replace b by $b\ell^{-1}$ mod N:

$$\widetilde{\theta}_\chi(iy) \;=\; \frac{-i}{N\sqrt{y}}\sum_{\ell\in\mathbb{Z}}\chi^{-1}(\ell)\, f\Big(\frac{\ell}{N\sqrt{y}}\Big)\sum_{b \bmod N}\chi(b)e^{2\pi ib/N}$$

The inner sum has become a Gauss sum $\langle\chi,\psi\rangle$ *not* depending on ℓ,

$$\langle\chi,\psi\rangle \;=\; \sum_{b \bmod N}\chi(b)\,\overline{\psi}(b) \qquad \text{(with } \psi(b) = e^{-2\pi ib/N})$$

so we have

$$\widetilde{\theta}_\chi(iy) \;=\; \frac{-i\langle\chi,\psi\rangle}{N\sqrt{y}}\cdot\sum_{\ell\in\mathbb{Z}}\chi^{-1}(\ell)\, f\Big(\frac{\ell}{N\sqrt{y}}\Big) \;=\; \frac{-i\langle\chi,\psi\rangle}{N\sqrt{y}}\cdot\widetilde{\theta}_{\chi^{-1}}\Big(\frac{i}{N^2y}\Big) \qquad (\text{odd } \chi \text{ primitive mod } N)$$

giving the functional equation

$$\widetilde{\theta}_\chi(iy) \;=\; \frac{-i\langle\chi,\psi\rangle}{N\sqrt{y}}\;\cdot\;\widetilde{\theta}_{\chi^{-1}}\Big(\frac{i}{N^2y}\Big) \qquad (\text{odd } \chi \text{ primitive mod } N)$$

[4] This Fourier transform is readily computed: view the integral defining the Fourier transform as a contour integral along the real axis. Moving the contour upward by $i\xi$ immediately computes the Fourier transform, much as was done with the Gaussian itself.

[2.3.1] Remark: The functional equation relates $\widetilde{\theta}_\chi$ to $\widetilde{\theta}_{\chi^{-1}}$, rather than to $\widetilde{\theta}_\chi$ itself.

[2.3.2] Remark: The *flip* in the argument y is $y \to 1/N^2 y$, not simply $y \to 1/y$. The fixed-point of this map is $1/N$, influencing the break-up of the integral in proof of the analytic continuation and functional equation, next.

[2.4] Analytic continuation and functional equation Starting from the integral representation,

$$\pi^{-(s+1)/2}\,\Gamma(\tfrac{s+1}{2})\,L(s,\chi) \;=\; \int_0^\infty y^{\frac{s}{2}}\,\frac{\widetilde{\theta}_\chi(iy)}{2}\,\frac{dy}{y} \;=\; \int_{1/N}^\infty y^{\frac{s}{2}}\,\frac{\widetilde{\theta}_\chi(iy)}{2}\,\frac{dy}{y} + \int_0^{1/N} y^{\frac{s}{2}}\,\frac{\widetilde{\theta}_\chi(iy)}{2}\,\frac{dy}{y}$$

As for $\zeta(s)$, the integral from $1/N$ to ∞ is nicely convergent for all $s \in \mathbb{C}$, and gives an entire function. The goal is to use the functional equation to convert the integral from 0 to $1/N$ to the other sort, and then symmetrize the resulting expression as much as possible. That is, replace y by $1/N^2 y$ in the problemmatical integral from 0 to $1/N$, obtaining

$$\tfrac{1}{2}\int_{1/N}^\infty (1/N^2 y)^{\frac{s}{2}}\,\widetilde{\theta}_\chi\Big(\frac{i}{N^2 y}\Big)\,\frac{dy}{y} \;=\; \tfrac{1}{2}\,N^{-s}\int_{1/N}^\infty y^{-\frac{s}{2}}\,\widetilde{\theta}_\chi\Big(\frac{i}{N^2 y}\Big)\,\frac{dy}{y}$$

Replacing χ by χ^{-1} in the functional equation above gives

$$\widetilde{\theta}_{\chi^{-1}}(iy) \;=\; \frac{-i\langle \chi^{-1},\psi\rangle}{N\sqrt{y}}\cdot \widetilde{\theta}_\chi(\frac{i}{N^2 y})$$

Rearranging,

$$\widetilde{\theta}_\chi(\frac{i}{N^2 y}) \;=\; \frac{-iN\sqrt{y}}{\langle \chi^{-1},\psi\rangle}\widetilde{\theta}_{\chi^{-1}}(iy)$$

Substituting this gives

$$\frac{-iN^{1-s}}{\langle \chi^{-1},\psi\rangle}\int_{1/N}^\infty y^{\frac{1-s}{2}}\,\frac{\widetilde{\theta}_{\chi^{-1}}(iy)}{2}\,\frac{dy}{y}$$

That is,

$$\pi^{-s/2}\,\Gamma(\tfrac{s}{2})\,L(s,\chi) \;=\; \int_{1/N}^\infty y^{\frac{s}{2}}\,\frac{\widetilde{\theta}_\chi(iy)}{2}\,\frac{dy}{y} \;+\; \frac{-iN^{1-s}}{\langle \chi^{-1},\psi\rangle}\int_{1/N}^\infty y^{\frac{1-s}{2}}\,\frac{\widetilde{\theta}_{\chi^{-1}}(iy)}{2}\,\frac{dy}{y}$$

This gives the *analytic continuation*, but is not symmetrized.

Recall that for χ *primitive*,

$$|\langle \chi,\psi\rangle| \;=\; \sqrt{N}$$

We do not pretend to know the *argument* of the Gauss sum, only its *size*: let

$$\varepsilon(\chi) \;=\; \frac{-i\sqrt{N}}{\langle \chi^{-1},\psi\rangle}$$

Then $|\varepsilon(\chi)| = 1$ and

$$\pi^{-s/2}\,\Gamma(\tfrac{s}{2})\,L(s,\chi) \;=\; \int_{1/N}^\infty \Big(y^{\frac{s}{2}}\,\frac{\widetilde{\theta}_\chi(iy)}{2} \;+\; \varepsilon(\chi)N^{\frac{1}{2}-s}y^{\frac{1-s}{2}}\,\frac{\widetilde{\theta}_{\chi^{-1}}(iy)}{2}\Big)\,\frac{dy}{y}$$

Ignoring $\varepsilon(\chi)$ for a moment, symmetrize the powers of the conductor N, by dividing through by $N^{s/2}$, giving

$$N^{-s/2}\,\pi^{-s/2}\,\Gamma(\tfrac{s}{2})\,L(s,\chi) \;=\; \int_{1/N}^\infty \Big((Ny)^{\frac{s}{2}}\,\frac{\widetilde{\theta}_\chi(iy)}{2} \;+\; \varepsilon(\chi)(Ny)^{\frac{1-s}{2}}\,\frac{\widetilde{\theta}_{\chi^{-1}}(iy)}{2}\Big)\,\frac{dy}{y}$$

Regarding $\varepsilon(\chi)$: since χ is *odd*,

$$\overline{\langle\chi^{-1},\psi\rangle} \;=\; \langle\chi,\overline{\psi}\rangle \;=\; \sum_b \chi(b)\overline{\psi}(-b) \;=\; \chi(-1)\sum_b \chi(b)\overline{\psi}(b) \;=\; \chi(-1)\langle\chi,\psi\rangle \;=\; -\langle\chi,\psi\rangle \qquad \text{(for odd } \chi)$$

Thus,

$$\varepsilon(\chi)\cdot\varepsilon(\chi^{-1}) \;=\; \frac{-i\sqrt{N}\cdot -i\sqrt{N}}{\langle\chi^{-1},\psi\rangle\cdot\langle\chi,\psi\rangle} \;=\; \frac{-N}{-\overline{\langle\chi,\psi\rangle}\cdot\langle\chi,\psi\rangle} \;=\; 1$$

giving the symmetry

$$\varepsilon(\chi)\cdot\varepsilon(\chi^{-1}) \;=\; 1 \qquad \text{(for } \chi \text{ odd)}$$

Thus, we have

$$N^{s/2}\,\pi^{-(s+1)/2}\,\Gamma(\tfrac{s+1}{2})\,L(s,\chi) \;=\; \int_{1/N}^\infty \Big((Ny)^{\frac{s}{2}}\,\frac{\widetilde{\theta}_\chi(iy)}{2} \;+\; \varepsilon(\chi)(Ny)^{\frac{1-s}{2}}\,\frac{\widetilde{\theta}_{\chi^{-1}}(iy)}{2}\Big)\,\frac{dy}{y}$$

$$=\; \varepsilon(\chi)\int_{1/N}^\infty \Big(\varepsilon(\chi^{-1}(Ny)^{\frac{s}{2}}\,\frac{\widetilde{\theta}_\chi(iy)}{2} \;+\; (Ny)^{\frac{1-s}{2}}\,\frac{\widetilde{\theta}_{\chi^{-1}}(iy)}{2}\Big)\,\frac{dy}{y}$$

$$=\; \varepsilon(\chi)\cdot N^{-(1-s)/2}\,\pi^{-(1-s)/2}\,\Gamma(\tfrac{1-s}{2})\,L(1-s,\chi^{-1}) \qquad (\chi \text{ odd})$$

giving the functional equation

$$N^{\frac{s}{2}}\,\pi^{-\frac{(s+1)}{2}}\,\Gamma(\tfrac{s+1}{2})\,L(s,\chi) \;=\; \varepsilon(\chi)\cdot N^{\frac{(1-s)}{2}}\,\pi^{-\frac{(2-s)}{2}}\,\Gamma(\tfrac{2-s}{2})\,L(1-s,\chi^{-1}) \qquad \text{(odd } \chi \text{ primitive mod } N)$$

where

$$\varepsilon(\chi) \;=\; \frac{-i\sqrt{N}}{\langle\chi^{-1},\psi\rangle} \qquad \text{and} \qquad |\varepsilon(\chi)| \;=\; 1 \qquad \text{(for } \chi \text{ odd, primitive modulo } N)$$

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3. *Dedekind zeta function* $\zeta_{\mathfrak{o}}(s)$ *for Gaussian integers* $\mathfrak{o}$

[3.1] Integral representation For $\mathfrak{o} = \mathbb{Z}[i]$, with $N\alpha = \alpha\overline{\alpha} = |\alpha|^2$, the zeta function

$$\zeta_{\mathfrak{o}}(s) \;=\; \sum_{0\neq\alpha\in\mathfrak{o} \bmod \mathfrak{o}^\times} \frac{1}{(N\alpha)^s} \;=\; \frac{1}{4}\sum_{m,n}\frac{1}{(m^2+n^2)^s} \qquad \text{(summed over } m,n \text{ not both 0)}$$

is readily expressed as an integral transform of the theta function

$$\theta(iy) \;=\; \sum_{m,n\in\mathbb{Z}} e^{-\pi(m^2+n^2)y} \;=\; \big(\sum_{n\in\mathbb{Z}} e^{-\pi n^2 y}\big)^2$$

by a computation similar to those above:

$$\int_0^\infty y^s\,\frac{\theta(iy)-1}{4}\,\frac{dy}{y} \;=\; \frac{1}{4}\sum_{m,n}\int_0^\infty y^s\,e^{-\pi(m^2+n^2)y}\,\frac{dy}{y} \;=\; \frac{1}{4}\sum_{m,n}\frac{1}{\big(\pi(m^2+n^2)\big)^s}\int_0^\infty y^s\,e^{-y}\,\frac{dy}{y}$$

$$=\; \pi^{-s}\,\Gamma(s)\,\zeta_{\mathfrak{o}}(s)$$

[3.2] Functional equation of θ The theta function is made from a Gaussian $f(u,v) = e^{-pi(u^2+v^2)}$ in two variables. The fact that this function is a product of two single-variable functions is useful on other occasions, but not essential at present.

The Poisson summation formula

$$\sum_{x\in\mathbb{Z}^2} \varphi(x) \;=\; \sum_{x\in\mathbb{Z}^2} \widehat{\varphi}(x)$$

will be applied to

$$\varphi(u,v) \;=\; f(\sqrt{y}\cdot u, \sqrt{y}\cdot v)$$

The Fourier transform of φ is readily computed

$$\widehat{\varphi}(u,v) \;=\; \frac{1}{y}\, f\big(\frac{u}{\sqrt{y}}, \frac{v}{\sqrt{y}}\big)$$

Thus,

$$\theta(iy) \;=\; \sum_{m,n\in\mathbb{Z}} e^{-\pi(m^2+n^2)y} \;=\; \sum_{x\in\mathbb{Z}^2} \frac{1}{y} e^{-\pi(m^2+n^2)/y} \;=\; \frac{1}{y}\,\theta\big(\frac{i}{y}\big)$$

[3.3] Analytic continuation and functional equation As in all previous cases, the integral in the integral representation breaks into two pieces, one already entire, and the other convertible to the same form via the functional equation of θ, with elementary leftovers describing the poles. That is, first,

$$\pi^{-s}\,\Gamma(s)\,\zeta_{\mathfrak{o}}(s) \;=\; \int_0^\infty y^s\,\frac{\theta(iy)-1}{4}\,\frac{dy}{y} \;=\; \int_1^\infty y^s\,\frac{\theta(iy)-1}{4}\,\frac{dy}{y} + \int_0^1 y^s\,\frac{\theta(iy)-1}{4}\,\frac{dy}{y}$$

and the integral over $[1,\infty)$ is entire. Use the functional equation and replacement of y by $1/y$ to obtain

$$\int_0^1 y^s\,\frac{\theta(iy)-1}{4}\,\frac{dy}{y} \;=\; \int_0^1 y^s\,\frac{\frac{1}{y}\theta(\frac{i}{y})-1}{4}\,\frac{dy}{y} \;=\; \int_1^\infty y^{-s}\,\frac{y\theta(iy)-1}{4}\,\frac{dy}{y}$$

$$= \int_1^\infty y^{-s}\,\frac{y(\theta(iy)-1)+y-1}{4}\,\frac{dy}{y} \;=\; \int_1^\infty y^{1-s}\,\frac{\theta(iy)-1}{4}\,\frac{dy}{y} + \frac{1}{4}\int_1^\infty y^{1-s} - y^{-s}\,\frac{dy}{y}$$

$$= \int_1^\infty y^{1-s}\,\frac{\theta(iy)-1}{4}\,\frac{dy}{y} + \frac{1}{4}\Big(\frac{1}{s-1} - \frac{1}{s}\Big)$$

Altogether,

$$\pi^{-s}\,\Gamma(s)\,\zeta_{\mathfrak{o}}(s) \;=\; \int_1^\infty \big(y^s + y^{1-s}\big)\,\frac{\theta(iy)-1}{4}\,\frac{dy}{y} + \frac{1}{4}\Big(\frac{1}{s-1} - \frac{1}{s}\Big)$$

This gives the meromorphic continuation and functional equation

$$\pi^{-s}\,\Gamma(s)\,\zeta_{\mathfrak{o}}(s) \;=\; \pi^{-(1-s)}\,\Gamma(1-s)\,\zeta_{\mathfrak{o}}(1-s)$$

The residues of the *completed* zeta function $\pi^{-s}\,\Gamma(s)\,\zeta_{\mathfrak{o}}(s)$ at $s=0,1$ are $1/4$. Thus, the residues of $\zeta_{\mathfrak{o}}(s)$ itself are $1/4$ at $s=0$, and $\pi/4$ at $s=1$.

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4. *Grossencharacter L-functions $L(s,\chi)$ for Gaussian integers*

Another kind of L-function appears for *every* number field larger than $\mathbb{Q}$, called *Hecke characters* or *größencharakteren*, anglicized to *grossencharacters.*

Rather than give a general definition, we consider examples for Gaussian integers $\mathfrak{o} = \mathbb{Z}[i]$. These characters and L-functions are essential to applications of harmonic analysis to number theory. For example, study of the distribution of Gaussian primes in *angular sectors* uses Hecke characters and L-functions in a style parallel to the use of Dirichlet characters and L-functions in the study of the distribution of primes modulo N. In one case, we have harmonic analysis on the *circle*, in the other case, harmonic analysis on $(\mathbb{Z}/N)^\times$.

[4.1] Hecke L-functions The simplest Hecke characters on $\mathbb{Z}[i]$ are the multiplicative group homomorphisms

$$\chi \;:\; \alpha \longrightarrow (\alpha/\overline{\alpha})^\ell \qquad\qquad (\ell \in \mathbb{Z}, \text{ group homomorphism } \mathbb{C}^\times \longrightarrow S^1)$$

For χ to be well-defined on *ideals* it must give the same value on every generator of ideals, which requires that χ be trivial on *units*: $\chi(\eta) = 1$ for every $\eta \in \mathfrak{o}^\times$. In this example, triviality on units is the condition

$$1 \;=\; \chi(i) \;=\; \Big(\frac{i}{-i}\Big)^\ell \;=\; (-1)^\ell$$

so $\ell \in 2\mathbb{Z}$. Let $N\alpha = \alpha\overline{\alpha} = |\alpha|^2$ as in the discussion of the zeta-function of the Gaussian integers. For χ trivial on units, there are the corresponding Hecke L-functions

$$L(s,\chi) \;=\; \sum_{0\neq\alpha\in\mathfrak{o} \bmod \mathfrak{o}^\times} \frac{\chi(\alpha)}{N\alpha^s} \;=\; \prod_{\varpi \text{ prime}} \frac{1}{1-\dfrac{\chi(\varpi)}{N\varpi^s}}$$

$$= \sum_{0\neq\alpha\in\mathfrak{o} \bmod \mathfrak{o}^\times} \frac{(\alpha/\overline{\alpha})^\ell}{N\alpha^s} \;=\; \prod_{\varpi \text{ prime}} \frac{1}{1-\dfrac{(\varpi/\overline{\varpi})^\ell}{N\varpi^s}} \qquad\qquad (\text{with } \chi(\alpha) = (\alpha/\overline{\alpha})^\ell, \text{ for } \ell \in 2\mathbb{Z})$$

[4.2] Harmonic theta series To obtain any L-function via an integral representation from a theta series, the theta series must mimic the formation of the L-function. For Hecke L-functions with $\chi(\alpha) = (\alpha/\overline{\alpha})^\ell$, we might contemplate something like

$$\theta_\chi(iy) \;=\; \sum_{m,n\in\mathbb{Z}} \Big(\frac{m+in}{m-in}\Big)^\ell\, e^{-\pi(m^2+n^2)y} \qquad\qquad (???)$$

This fails: although the function

$$(u,v) \;\longrightarrow\; \Big(\frac{u+iv}{u-iv}\Big)^\ell\, e^{-\pi(u^2+v^2)}$$

has the right shape to produce the L-function by an integral, this function is *not* a Schwartz function, so Poisson summation does not easily apply. After some experimentation, one might try something like

$$\theta_\chi(iy) \;=\; \sum_{m,n\in\mathbb{Z}} (m+in)^\ell\, e^{-\pi(m^2+n^2)y} \qquad\qquad (???\ \ (\text{for } \ell \geq 0))$$

As with the *odd* Dirichlet characters, to apply Poisson summation, we might want the theta function to be of the form

$$\theta(iy) \;=\; \sum_{v\in\mathbb{Z}^2} \varphi(v \cdot \sqrt{y})$$

If so, then we'd take, instead,

$$\theta_\chi(iy) \;=\; \sum_{m,n\in\mathbb{Z}} (m+in)^\ell\, y^{\ell/2}\, e^{-\pi(m^2+n^2)y} \qquad\qquad (???\ \ (\text{for } \ell \geq 0))$$

That is, there would be an additional factor of $y^{\ell/2}$. Both conventions, with and without the factor of $y^{e\ll/2}$, make sense. For immediate purposes, we *include* this factor. For $\ell \geq 0$, the function $(u+iv)^\ell e^{-\pi(u^2+v^2)}$ is

Schwartz, but an adaptation is needed for $\ell < 0$. Further, noting that the exponents of $m + in$ and $m - in$ should differ by ℓ, we anticipate that probably a good choice is [5]

$$\theta_\chi(iy) \;=\; \begin{cases} \displaystyle\sum_{m,n\in\mathbb{Z}} (m+in)^{2\ell}\, y^\ell\, e^{-\pi(m^2+n^2)y} & (\text{for } \ell \geq 0) \\[2ex] \displaystyle\sum_{m,n\in\mathbb{Z}} (m-in)^{-2\ell}\, y^\ell\, e^{-\pi(m^2+n^2)y} & (\text{for } \ell \leq 0) \end{cases}$$

That is, the exponent in the polynomial part is 2ℓ, not ℓ, even though ℓ is already *even*. The polynomials $(u, v) \to (u + iv)^\ell$ for $\ell \geq 0$ and $(u, v) \to (u - iv)^{-\ell}$ for $\ell \leq 0$ are *holomorphic* on $\mathbb{R}^2 \approx \mathbb{C}$. Thus, they are *harmonic*. [6] For this reason, these theta series are called *harmonic* theta series. The functional equations will be proven via Poisson summation, below.

[4.3] Integral representation It is reasonable to see what happens when we write a simple integral involving the theta functions. Take $\ell > 0$ for simplicity. Then the lattice point $(0, 0)$ does not contribute to the sum, and no subtraction is necessary, unlike the case $\ell = 0$ arising for the zeta function of $\mathfrak{o}$ itself. Since $|\mathfrak{o}^\times| = 4$, divide through by 4, as in the case of the zeta function:

$$\int_0^\infty y^s\, \frac{\theta_\chi(iy)}{4}\, \frac{dy}{y} \;=\; \frac{1}{4}\sum_{m,n}(m+in)^{2\ell} \int_0^\infty y^s\, y^\ell\, e^{-\pi(m^2+n^2)y}\, \frac{dy}{y}$$

$$= \frac{1}{4}\sum_{m,n}(m+in)^{2\ell}\cdot \frac{1}{\pi^s}\, \frac{1}{(m^2+n^2)^{s+\ell}} \int_0^\infty y^{s+\ell}\, e^{-y}\, \frac{dy}{y}$$

$$= \pi^{-(s+\ell)}\,\Gamma(s+\ell)\, \frac{1}{4}\sum_{m,n}(m+in)^{2\ell}\cdot\frac{1}{(m^2+n^2)^{s+\ell}} \;=\; \frac{\pi^{-(s+\ell)}\,\Gamma(s+\ell)}{4} \sum_{m,n} \frac{[(m+in)/(m-in)]^\ell}{(m^2+n^2)^s}$$

$$= \pi^{-(s+\ell)}\,\Gamma(s+\ell)\, L(s,\chi) \qquad\qquad (\text{for } \chi(\alpha) = (\alpha/\overline{\alpha})^\ell \text{ and } 0 < \ell \in 2\mathbb{Z})$$

Note that *omission* of the factor y^ℓ in $\theta_\chi(iy)$ would yield $L(s - \ell, \chi)$ rather than $L(s, \chi)$. To have confidence that the latter expression is a good normalization for the L-function, we need to see the theta-function's functional equation and the consequent functional equation for $L(s, \chi)$.

[4.4] Poisson summation, Hecke's identity, functional equation of θ_χ Letting

$$\varphi(u, v) \;=\; (u + iv)^{2\ell}\, e^{-\pi(u^2+v^2)} \qquad\qquad (\text{for } 0 < \ell \in 2\mathbb{Z})$$

the theta function is exactly the Schwartz function φ, dilated by $\sqrt{y}$, summed over a lattice:

$$\theta_\chi(iy) \;=\; \sum_{m,n\in\mathbb{Z}} \varphi\big((m,n)\cdot\sqrt{y}\big)$$

[5] Regarding the question of including an extra factor of y^ℓ, or not: when the extra factor is *omitted*, then $\theta_\chi(z)$ is a holomorphic function in $z \in \mathfrak{H}$. This is one argument in favor of dropping that factor.

[6] A twice-differentiable function f on $\mathbb{R}^n$ is *harmonic* if it is annihilated by the *Laplacian* $\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$. In two variables, the Laplacian factors

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \;=\; \Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)$$

The second linear differential operator is the *Cauchy-Riemann* operator. Annihilation by the Cauchy-Riemann operator is equivalent to *holomorphy*.

Poisson summation gives

$$\theta_\chi(iy) \;=\; \sum_{m,n\in\mathbb{Z}} \varphi\big((m,n)\cdot\sqrt{y}\big) \;=\; \frac{1}{y}\sum_{m,n\in\mathbb{Z}} \widehat{\varphi}\big((m,n)\cdot\frac{1}{\sqrt{y}}\big)$$

To compute the Fourier transform $\widehat{\varphi}$, use coordinates $w = u+iv$ and $\overline{w} = u - iv$. The Fourier transform is

$$\widehat{\varphi}(u,v) \;=\; \int_{\mathbb{R}^2} e^{-2\pi i(u\xi+v\eta)}\,\varphi(\xi,\eta)\,d\xi\,d\eta \;=\; \int_{\mathbb{R}^2} e^{-\pi i(w\overline{z}+\overline{w}z)}\,z^{2\ell}\,e^{-\pi z\overline{z}}\,d\xi\,d\eta \qquad \text{(with } z=\xi+i\eta)$$

Now use

$$\frac{\partial}{\partial w} \;=\; \tfrac{1}{2}\Big(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\Big) \qquad\qquad \frac{\partial}{\partial \overline{w}} \;=\; \tfrac{1}{2}\Big(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\Big)$$

and compute

$$\int_{\mathbb{R}^2} e^{-\pi i(w\overline{z}+\overline{w}z)}\,z^{2\ell}\,e^{-\pi z\overline{z}}\,d\xi\,d\eta \;=\; \frac{1}{(-\pi i)^{2\ell}}\Big(\frac{\partial}{\partial\overline{w}}\Big)^{2\ell}\int_{\mathbb{R}^2} e^{-\pi i(w\overline{z}+\overline{w}z)}\,e^{-\pi z\overline{z}}\,d\xi\,d\eta$$

$$=\; \frac{1}{(-\pi i)^{2\ell}}\Big(\frac{\partial}{\partial\overline{w}}\Big)^{2\ell} e^{-\pi w\overline{w}} \;=\; \frac{1}{(-\pi i)^{2\ell}}(-\pi w)^{2\ell}\,e^{-\pi w\overline{w}} \;=\; i^{-2\ell}\,w^{2\ell}\,e^{-\pi w\overline{w}}$$

since the Gaussian $e^{-\pi z\overline{z}}$ is its own Fourier transform. That is, $\widehat{\varphi} = i^{-2\ell}\varphi$. That is,

$$\text{Fourier transform}\,\big((u+iv)^{2\ell}\,e^{-\pi(u^2+v^2)}\big) \;=\; (-1)^{-\ell}\cdot(u+iv)^{2\ell}\,e^{-\pi(u^2+v^2)}$$

By Poisson summation,

$$\theta_\chi(iy) \;=\; \sum_{m,n\in\mathbb{Z}} \varphi\big((m,n)\cdot\sqrt{y}\big) \;=\; \frac{(-1)^\ell}{y}\cdot\sum_{m,n\in\mathbb{Z}} \varphi\big((m,n)\cdot\frac{1}{\sqrt{y}}\big) \;=\; \frac{(-1)^\ell}{y}\cdot\theta_\chi\big(\frac{1}{y}\big)$$

[4.4.1] Remark: If the factor of y^ℓ were *omitted* the functional equation of the corresponding theta function would be

$$\sum_{m,n\in\mathbb{Z}} (m+in)^{2\ell}\,e^{-\pi(m^2+n^2)y} \;=\; \frac{(-1)^\ell}{y^{1+2\ell}}\sum_{m,n\in\mathbb{Z}} (m+in)^{2\ell}\,e^{-\pi(m^2+n^2)/y}$$

This can be obtained from the result above by dividing through by y^ℓ and rearranging slightly: these two forms of the functional equation are *not* truly different.

[4.5] Analytic continuation and functional equation

As in all other examples, break the integral of the integral representation into two pieces, one of which is entire, the other converted to the same sort of entire integral by using the functional equation of θ_χ. Treat just the case $0 < \ell \in 2\mathbb{Z}$.

$$\pi^{-(s+\ell)}\,\Gamma(s+\ell)\,L(s,\chi) \;=\; \int_0^\infty y^s\,\frac{\theta_\chi(iy)}{4}\,\frac{dy}{y} \;=\; \int_1^\infty y^s\,\frac{\theta_\chi(iy)}{4}\,\frac{dy}{y} + \int_0^1 y^s\,\frac{\theta_\chi(iy)}{4}\,\frac{dy}{y}$$

The integral on $[1,\infty$ is entire in s. The integral on $[0,1]$ is transformed via the functional equation of θ_χ and replacing y by $1/y$. Since $\ell > 0$, the theta function has no constant term to subtract, so the book-keeping is simpler:

$$\int_0^1 y^s\,\frac{\theta_\chi(iy)}{4}\,\frac{dy}{y} \;=\; (-1)^\ell\int_0^1 y^s\,\frac{\frac{1}{y}\theta_\chi(\frac{i}{y})}{4}\,\frac{dy}{y} \;=\; (-1)^\ell\int_1^\infty y^{1-s}\,\frac{\theta_\chi(iy)}{4}\,\frac{dy}{y}$$

That is,

$$\pi^{-(s+\ell)}\,\Gamma(s+\ell)\,L(s,\chi) \;=\; \int_1^\infty \big(y^s + (-1)^\ell y^{1-s}\big)\,\frac{\theta_\chi(iy)}{4}\,\frac{dy}{y}$$

The right-hand side is entire, and the functional equation gives a symmetry under $s \leftrightarrow 1-s$:

$$\pi^{-(s+\ell)}\,\Gamma(s+\ell)\,L(s,\chi) \;=\; (-1)^\ell \cdot \pi^{-(1-s+\ell)}\,\Gamma(1-s+\ell)\,L(1-s,\chi) \qquad \text{(for } 0<\ell\in 2\mathbb{Z})$$

The computation for $\chi(\alpha) = (\alpha/\overline{\alpha})^\ell$ with $\ell < 0$ is similar. The two computations can be put into a common form

$$\pi^{-(s+|\ell|)}\,\Gamma(s+|\ell|)\,L(s,\chi) \;=\; (-1)^\ell \cdot \pi^{-(1-s+|\ell|)}\,\Gamma(1-s+|\ell|)\,L(1-s,\chi) \qquad \text{(for } \ell\in 2\mathbb{Z})$$

[4.6] Where's the contragredient? For Dirichlet L-functions, the functional equation relates $L(s,\chi)$ and $L(1-s,\overline{\chi})$, with the complex conjugate character $\overline{\chi}$ rather than χ. That was the simplest example of the appearance of a *contragredient*. Where is the contragredient in the functional equation for the Hecke character L-functions above? The answer is that, although the Hecke character $\chi(\alpha) = (\alpha/\overline{\alpha})^\ell$ and its conjugate $\overline{\chi}(\alpha) = (\alpha/\overline{\alpha})^{-\ell}$ are obviously distinct, their L-functions are *equal*. This can be seen from the defining sum, by replacing α by $\overline{\alpha}$:

$$L(s,\chi) \;=\; \sum_{0\neq\alpha\in\mathfrak{o} \bmod \mathfrak{o}^\times} \frac{(\alpha/\overline{\alpha})^\ell}{|N\alpha|^s} \;=\; \sum_{0\neq\alpha\in\mathfrak{o} \bmod \mathfrak{o}^\times} \frac{(\overline{\alpha}/\alpha)^\ell}{|N\overline{\alpha}|^s} \;=\; \sum_{0\neq\alpha\in\mathfrak{o} \bmod \mathfrak{o}^\times} \frac{(\alpha/\overline{\alpha})^{-\ell}}{|N\alpha|^s} \;=\; L(s,\overline{\chi})$$

Thus, here, the functional equation *does* relate $L(s,\chi)$ and $L(1-s,\overline{\chi})$, but, also, $L(s,\chi) = L(s,\overline{\chi})$.

For L-functions *combining* congruence conditions with Hecke-character features as above, the L-function of the contragredient would rarely match the original.