



- [1.4] Let  $Y$  be a subset of a topological space  $X$ , with inclusion map  $i : Y \rightarrow X$ . The mapping-property definition of the **subset topology** is that, given a continuous map  $f : Z \rightarrow X$  of a topological space  $Z$  to  $X$ , with image  $f(Z)$  contained in  $i(Y)$ , there is a unique continuous map  $F : Z \rightarrow Y$  such that  $f = i \circ F$ . In a diagram,

$$\begin{array}{ccc} & & Z \\ & \swarrow & \downarrow f \\ Y & \xrightarrow{i} & X \end{array}$$

Prove that this diagrammatic definition is fulfilled by the usual *construction* of the subset topology.

- [1.5] By definition, a **free group on a set**  $S$  is a group  $G$  and a set map  $i : S \rightarrow G$  such that, for every group  $H$  and set map  $f : S \rightarrow H$  there is a unique *group homomorphism*  $F : G \rightarrow H$  such that  $f = F \circ i$ , that is, such that we have a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{F} & H \\ \uparrow i & \nearrow f & \\ S & & \end{array}$$

Prove that, as usual, there is at most one free group on a given set  $S$ . Prove that the map  $i : S \rightarrow G$  is *injective* as a set map.

- [1.6] Prove that the following construction of a free group  $i : S \rightarrow G$  on a *finite* set succeeds. [4] First, show that for any set map  $f : S \rightarrow H$ , the subgroup  $\langle f(S) \rangle$  of  $H$  generated by [5] the image  $f(S)$  is *countable* (either finite or countable infinite). Show that there are finitely-many (isomorphism classes of) groups of a given finite cardinality  $n$ . Show that the collection of (isomorphism classes of) countably-infinite groups is no larger than the power set of a countable set. Then show that the collection of *isomorphism classes of set maps*  $j : S \rightarrow G$  with  $G$  a group generated by  $j(S)$  is a *set*  $I$ . Then let  $G = \prod_{H \in I} H$  be the group product over (representatives [6] for) isomorphism classes  $f_H : S \rightarrow H$  in  $I$ , with  $j : S \rightarrow G$  defined by

$$\text{pr}_H \circ j = f_H$$

where  $\text{pr}_H : G \rightarrow H$  is the projection for  $H \in I$ . Then show that this  $j : S \rightarrow G$  is a free group on  $S$ .

- [1.7] Prove that the not-necessarily-abelian group coproduct of two groups  $G$  and  $H$  exists, as a quotient of a suitable free group.
- [1.8] Let  $A_i$  ( $i$  in an index set  $I$ ) be abelian groups, and describe the natural map from the set coproduct  $\coprod_i^{\text{sets}} A_i$  to the abelian group coproduct  $\coprod_i^{\text{ab grps}} A_i$ , and observe that it is very rarely a bijection (unless the index set has just one element, or maybe playing on coincidences like  $2 + 2 = 2 \cdot 2$ ).
- [1.9] Why is it *not* paradoxical that the underlying set of a (abelian-group) coproduct of abelian groups is *not* the disjoint union (set coproduct) of the underlying sets?

[4] A version of this appears in Lang's *Algebra*, and is attributed to Jacques Tits. The same approach succeeds for arbitrary sets  $S$  with a few additional complications.

[5] A usual, the subgroup *generated by* a subset is the intersection of all subgroups containing that subset.

[6] Yes, a choice of representatives uses the Axiom of Choice in some form. The fact that this lends an air of indeterminacy to the construction is merely an artifact, though, since the uniqueness of a free group, if it exists at all, is already clear.

[1.10] Prove that the 2-solenoid  $\lim_n \mathbb{R}/2^n\mathbb{Z}$  is *not* homeomorphic to a product  $\mathbb{R}/\mathbb{Z} \times \mathbb{Z}_2$  (where  $\mathbb{Z}_2 = \lim_n \mathbb{Z}/2^n$ ).