1. Lake Ore-be-gone is a artificial lake outside of Gilbert Minnesota, formed by flooding an rectangular open pit iron mine. It is 250 meters long, 500 meters wide, and 125 meters deep.

(a) Use a triple integral to find the volume of water in the lake.

(b) If the density of iron ore (in kilograms per cubic meter) at the point \((x, y, z)\) is given by the function
\[
\delta(x, y, z) = \frac{3x + 4y + z}{100},
\]
compute how much ore was extracted from the mine.

(c) If the cost (in cents) to excavate a cubic meter of earth depends on the depth below the surface, and is given by:
\[
c(x, y, z) = z^2
\]
find the total cost of excavating the hole for lake Ore-be-Gone.

(d) Assuming that iron can always be sold for $4.46/kg, how much profit was made by selling the copper excavated from lake Ore-be-Gone.

(e) how would that revenue have changed if the mind had been excavated to a depth of 150 meters?

Solution:

(a) It is not strictly speaking necessary to use a triple integral to compute the volume of the lake, volume of a rectangular region is simply length*width*depth. In general we can always find the volume by integrating the constant function 1.

\[
\int_{0}^{250} \int_{0}^{500} \int_{0}^{125} 1 \, dz \, dy \, dx = 15625000.
\]

While it does not make a difference in the computation here, notice that I have setup the problem to use \(x\) for length, \(y\) for width, and \(z\) for depth. If you setup the problem another way, subsequent answers could be different from my own.

(b) We can integrate density in order to find mass. We will compute an integral over the same region, with a new integrand.

\[
\int_{0}^{250} \int_{0}^{500} \int_{0}^{125} \delta(x, y, z) \, dz \, dy \, dx = \int_{0}^{250} \int_{0}^{500} \int_{0}^{125} \frac{3x + 4y + z}{100} \, dz \, dy \, dx = 205078125.
\]

(c) In order to find the total cost of excavating lake ore-be-gone, we must integrate the cost function.

\[
\int_{0}^{250} \int_{0}^{500} \int_{0}^{125} z^2 \, dz \, dy \, dx = 81380200000
\]
In order to compare numbers, we will convert this to dollars by dividing by 100. In dollars, the excitation cost is 813802000.

(d) We now know the cost of extracting the iron ore from the mine, as well as the mass of iron extracted. Knowing that ore sells for 4.46/kg we can compute that there was a revune of $4.46 \times 205078124 = 914648000$, and a total profit of $1108030000$.

(e) Neither the density function nor cost function change when the depth increases, so we can again integrate density and cost in order to find profit. The only think to change is the bounds on the integrals. So to find cost and revunue we compute

\[ \int_{0}^{250} \int_{0}^{500} \int_{0}^{150} \frac{3x + 4y + z}{100} \, dz \, dy \, dx - \int_{0}^{250} \int_{0}^{500} \int_{0}^{150} z^2 \, dz \, dy \, dx = -298219000 \]

So extracting the additional ore would have cost more than the revunue from selling the ore.

2. Let $R$ be the region in $\mathbb{R}^2$ inside the parallelogram spanned by the vectors $(2, 2)$ and $(1, 0)$.

(a) For any $x \in [0, 3]$ find the possible values of $y$ (note, the $y$ value may depend on $x$).

(b) For any $y \in [0, 2]$ find the possible values of $x$ (note, he $x$ value may depend on $y$).

(c) Setup and evaluate an integral to compute the area of $R$. Choose your order of integration to make the evaluation easier

(d) Find the area of $R$ using geometry.

\textbf{Solution:}

(a) For a fixed $x \in [0, 3]$ we can find bounds on $y$, but those bounds will depend on $x$ and there is no single equation we can use to express the bounds on $y$. As we see in the picture
the bounds on \( y \) can be dictated by one of three lines and which line to use depends on the \( x \) value. It breaks up into three cases: \( 0 \leq x \leq 1 \), \( 1 \leq x \leq 2 \), and \( 2 \leq x \leq 3 \). So we have three sets of bounds:

- \( 0 \leq y \leq x \) when \( 0 \leq x \leq 1 \)
- \( x - 1 \leq y \leq x \) when \( 1 \leq x \leq 2 \)
- \( x \leq y \leq 2 \) when \( 2 \leq x \leq 3 \)

(b) For a fixed \( y \in [0, 2] \) we see that the smallest value of \( x \) is on the line \( x = y \) and the largest value is on the line \( x = y + 1 \), so the inequality bounds we find are

\[
0 \leq y \leq 2 \\
y \leq x \leq y + 1
\]

(c) With \( y \in [0, 2] \) the bounds were given by well behaved functions, so we will use \( dx \, dy \) as our order of integration, and setup the integral

\[
\int_{0}^{\frac{y+1}{2}} \int_{y}^{y+1} 1 \, dx \, dy.
\]

Evaluating we obtain

\[
\int_{0}^{\frac{y+1}{2}} \int_{y}^{y+1} 1 \, dx \, dy = \int_{0}^{\frac{y+1}{2}} x \bigg|_{y}^{y+1} \, dy = \int_{0}^{2} 1 \, dy = 2
\]

(d) The parallelogram is a 2-d parallelepiped, so its area a determinant. This allows us to compute the area as,

\[
\det \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} = 2.
\]

3. For the following regions \( R \), setup inequality bounds for the region, then compute the specified integral.

(a) \( R \) is the rectangle with vertices \((-2, 2), (0, 2), (0, 0), (-2, 0)\)

\[
\iint_{R} (3x^2 + 2y^3) \, dA
\]
Solution:

(a) We would usually draw a picture of the region here, but this region is very simple, so there is no need. We can see from the points that \( x \) varies freely between \(-2\) and 0, while \( y \) varies freely between 0 and 2. This gives the setup for the integral

\[
\int_{-2}^{0} \int_{0}^{2} (3x^2 + 2y^3) \, dx \, dy
\]

Which we can compute

\[
\int_{-2}^{0} \int_{0}^{2} (3x^2 + 2y^3) \, dx \, dy = \int_{0}^{2} \left( x^3 + 2y^3x \right)_{-2}^{0} \, dy \, dx = \\
= \int_{0}^{2} (8 + 4y^3) \, dy \, dx = \\
= 8y + y^4 \bigg|_{0}^{2} = 32
\]

(b) First we setup the bound on this integral. The easiest way to do this is to draw a picture.
We see that the bounds on $y$ vary between between $y = 0$ and $y = 5$. Further we see that for any $y$, $x$ can vary between $y^2$ and $5x$. This gives us all the information we need to setup the integral.

$$\int_0^5 \int_{y^2}^{5y} ye^x \, dx \, dy$$

Now we can compute this integral

$$\int_0^5 \int_{y^2}^{5y} ye^x \, dx \, dy = \int_0^5 ye^x \bigg|_{y^2}^{5y} =$$

$$= \int_0^5 ye^{5y} - ye^{y^2} =$$

$$= \left( \frac{ye^{5y}}{5} - \frac{e^{5y}}{25} - \frac{e^{y^2}}{2} \right) \bigg|_0^5 =$$

$$= \frac{27 + 23e^{25}}{50}$$

4. Consider the integral: $\int_0^1 \int_y^1 x^2 \, dx \, dy$

(a) Sketch the region of integration.
(b) Compute the Integral.
(c) Change the order of integration.
(d) Compute the new integral.

**Solution:**

(a)
(b) Computing this integral as given is straightforward.

\[
\int_0^1 \int_y^1 x^2 \, dy \, dx = \int_0^1 \frac{1}{3} x^3 y \bigg|_y^1 \, dy \\
= \int_0^1 \frac{1}{3} - \frac{1}{3} y^3 \, dy \\
= \frac{1}{3} y - \frac{1}{12} y^4 \bigg|_0^1 \\
= \frac{1}{4}
\]

(c) Reversing the order of integration we look now want to setup the integral is taken \( dydx \). So we look along the \( x \) axis and see that our integral will still go between 0 and 1. Then we look along the \( y \) axis, and for any \( x \in (0, 1) \) we know \( 0 \leq y \leq x \). This gives us all the information we need to setup this integral.

\[
\int_0^1 \int_0^x x^2 \, dy \, dx 
\]

(d) This computation is even easier.

\[
\int_0^1 \int_0^x x^2 \, dy \, dx = \int_0^1 x^2 y \bigg|_0^x \\
= \int_0^1 x^3 \\
= \frac{1}{4} x^4 \bigg|_0^1 \\
= \frac{1}{4}
\]

The fact that the answers agree should come as no surprise.

5. Consider the integral: \( \int_0^1 \int_0^{\sqrt{1-x^2}} 3 \, dy \, dx \)
(a) Sketch the region you are integrating over.

(b) Compute the Integral.

(c) Change the order of integration.

**Solution:**

(a)

(b) This integral is very similar to the integrals we computed in Lab 3, only easier.

\[
\int_{0}^{1} \int_{0}^{\sqrt{1-x^2}} 3\, dx\, dy = 3 \int_{0}^{1} \sqrt{1-x^2} \, dx \\
= 3 \int_{0}^{1} \arcsin(x) \, dx \\
= \sqrt{1-x^2} + x \arcsin(x) \bigg|_{0}^{1} = \frac{3\pi}{4}
\]

(c) This is something of a trick question. In this problem you can reverse the order of integration and have the integral

\[
\int_{0}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3\, dx\, dy
\]

(d) Compute the new integral.

Since changing the order of integration did not change the integral there is no need to recompute the new integral.
6. Consider the integral: $\int_0^9 \int_{\sqrt{y}}^3 (x + y) \, dx \, dy$

(a) Sketch the region you are integrating over.
(b) Compute the Integral.
(c) Change the order of integration.
(d) Compute the new integral.

Solution:

(a) [Diagram of the region of integration]

(b) $\int_0^9 \int_{\sqrt{y}}^3 (x + y) \, dx \, dy = \int_0^9 \left[ \frac{x^2}{2} + xy \right]_{\sqrt{y}}^3 \, dy$

$= \int_0^9 \left( \frac{9}{2} + \frac{9}{2} - \frac{5y}{2} - y^{3/2} \right) \, dy$

$= \left[ \frac{9}{2} \left( 9y + \frac{5y^2}{2} - \frac{4y^{5/2}}{5} \right) \right]_0^9 = \frac{891}{20}$

(c) $\int_0^3 \int_0^{x^2} (x + y) \, dy \, dx$

(d) $\int_0^3 \int_0^{x^2} (x + y) \, dy \, dx = \int_0^3 xy + \frac{x^2}{2} \left[ x^2 \right]_0^3 \, dx$

$= \int_0^3 x^3 + \frac{x^4}{2} \, dx$

$= \frac{1}{20} \left( x^4 (5 + 2x) \right) \bigg|_0^3 = \frac{891}{20}$
And although it should not really be a surprise, the two integrals are equal.

7. Setup and evaluate the integral over the region $V$ bounded by a cube, centered at $(0, 1, 2)$ on which all edges have length 2.

$$\iiint_{V} xyz \, dV$$

**Solution:** One way to setup this integral is

$$\int_{0}^{2} \int_{-1}^{1} \int_{1}^{3} xyz \, dx \, dz \, dy = \int_{0}^{2} \int_{-1}^{1} \frac{yz^2}{2} \, dz \, dy = \int_{0}^{2} 0 \, dy = 0$$

So we find the value of the integral to be zero with no further work. Of course, we didn’t have to evaluate $dxdzdy$, we could have used any order of integration just as easily.

8. Evaluate the triple integral. $\iiint_{-1}^{2} \int_{1}^{2} \int_{0}^{3} 3yz^2 \, dx \, dy \, dz$

**Solution:**

$$\int_{-1}^{2} \int_{1}^{2} \int_{0}^{3} 3yz^2 \, dx \, dy \, dz =$$

$$= 3 \int_{-1}^{2} \int_{0}^{3} y^2z^2 \, dy \, dz =$$

$$= 3 \int_{-1}^{2} y^2z^2 \, dy \, dz =$$

$$= 3 \left[ \frac{y^3z^2}{3} + \frac{yz^3}{2} \right]_{1}^{2} \, dz =$$

$$= 3 \left[ \frac{y^3z^2}{3} + \frac{yz^3}{2} \right]_{1}^{2} \, dz =$$

$$= 3 \left[ \frac{z^8}{3} + \frac{z^7}{2} - \frac{z^6}{3} - \frac{z^3}{2} \right]_{1}^{2} \, dz =$$

$$= 3 \left( \frac{z^9}{27} + \frac{z^8}{16} - \frac{z^6}{9} - \frac{z^3}{8} \right)_{-1}^{2} = \frac{1539}{16}$$
So we have compute the value of this integral to be

\[
\frac{1539}{16} \approx 96.1875
\]