1. Lily the ladybug, Bob the beetle, and Sid the Spider all live happily in the $xy$ plane. At time $t$ Bobby’s position is given by the parametric equation

$$b(t) = \begin{pmatrix} \cos(\pi t) \\ \sin(\pi t) \end{pmatrix}$$

Simultaneously, Lily’s position is given by

$$l(t) = \begin{pmatrix} t - 1 \\ t \end{pmatrix}$$

(a) Do the path of Bob and Lily ever cross paths? Do Bob and Lily ever meet?

(b) At time $t = 0$ Sid is located at the point $(-3, -2)$ and begins running in a straight line towards the point $(-1, 0)$, arriving at time $t = 1$. Find a parametric equation $s(t)$ for Sid’s path.

(c) Does Sid cross paths with Bob or Lily? Does Sid ever run into Bob or Lily?

**Solution:**

(a) Bob is travelling in a circle of radius one centered at the origin. Lily is travelling along the line $y = x + 1$. These paths intersect at the points $(-1, 0)$ and $(0, 1)$. In order for Bob and Lily to meet, they must both reach one of these points at the same time $t$. Lily is at the point $(-1, 0)$ when $t = 0$, but Bob doesn’t arrive there until $t = 1$. Lily arrives at the point $(0, 1)$ when $t = 1$, but Bob passed that point when $t = 1/2$. Even though Bob and Lily’s paths intersect, the bugs never meet.

(b) To find a parametric equation for Sid’s path, we can use the general equation for a line connecting points $a$ and $b$,

$$s(t) = a + t(b - a) = (1 - t)a + t b.$$  

In this equation, $l(0) = a$ and $l(1) = b$, so we can use $a = (-3, -2)$ and $b = (-1, 0)$. This gives the equation

$$s(t) = (1 - t)(-3, -2) + t(-2, 0) = (2t - 3, 2t - 2).$$

(c) Start by finding an equation for Sid’s path. The direction vector for Sid’s path is $(5/2, 5/2)$, so the slope is

$$m = \frac{\text{rise}}{\text{run}} = \frac{5/2}{5/2} = 1.$$  

Since Sid’s path goes through the point $(-3, -2)$, it must have equation

$$y + 1 = 1(x + 2)$$
This is identical to the path that Lily follows! To check whether Sid and Lily ever meet, we’ll check for a \( t \)-value where the \( x \)-coordinates for Sid’s and Lily’s paths agree, that is, \( t - 1 = -2t - 3t \). This occurs at \( t = 2 \), so Sid and Lily meet at the point \((-1,0)\) at time \( t = 2 \). As above, Sid and Bob’s paths intersect at \((-1,0)\) and \((0,1)\). Sid reaches \((-1,0)\) when \( t = 1 \), which is the same instance that Bob does. Their paths again cross at \((0,1)\) but Sid is at \((0,\frac{3}{2})\) when Bob is at the point \((0,-1)\).

2. Use function \( f(x,y) = x^2y + y^2 \) for the following exercises.

(a) Using the definition of the partial derivative, find \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \).

(b) Double check your answer by computing the \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) using the derivative rules.

Keeping in mind that when you compute \( \frac{\partial f}{\partial x} \) you view \( y \) as a constant.

(c) Find the \( x = c \) and \( y = c \) cross sections of the graph of \( f(x,y) \) for a constant \( c \).

Both cross sections are the graphs of function \( \mathbb{R} \to \mathbb{R} \), find the derivatives of both functions.

(d) The \( y = x \) cross section of \( f(x,y) \) is the graph of a function \( g(x) \).

**Solution:**

(a) Recall that the partial derivatives are defined using limits.

\[
\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h},
\]

\[
\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}.
\]

Using these definitions we have

\[
\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{(x+h)^2y + y^2 - (x^2y + y^2)}{h}
\]

\[
= \lim_{h \to 0} \frac{x^2y + 2xhy + h^2y + y^2 - x^2y - y^2}{h}
\]

\[
= \lim_{h \to 0} \frac{2xhy + h^2y}{h}
\]

\[
= \lim_{h \to 0} \frac{h(2xy + hy)}{h}
\]

\[
= \lim_{h \to 0} 2xy + hy
\]

\[
= 2xy.
\]

\[
\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{x^2(y+h) + (y+h)^2 - (x^2y + y^2)}{h}
\]

\[
= \lim_{h \to 0} \frac{x^2y + x^2h + y^2 + 2yh + h^2 - x^2y - y^2}{h}
\]

\[
= \lim_{h \to 0} \frac{x^2h + 2yh + h^2}{h}
\]

\[
= \lim_{h \to 0} x^2 + 2yh + h.
\]
(b) It is simple to check that these derivatives match the derivatives we would get by simply keeping one variable constant.

\[
\frac{\partial f}{\partial x} = 2xy, \\
\frac{\partial f}{\partial y} = x^2 + 2y.
\]

(c) The constant \( x \) and \( y \) cross sections are found by simply putting \( c \) for the appropriate variable.

The \( x = c \) cross section is

\[
f(c, y) = c^2 y + y^2
\]

The derivative of this with respect to \( y \) is

\[
\frac{df}{dy} = c^2 + 2y
\]

Which matches \( \frac{\partial f}{\partial y}(c, y) \).

The \( x = c \) cross section is

\[
h(x, c) = x^2 c + c^2
\]

The derivative of this with respect to \( x \) is

\[
\frac{df}{dy} = 2cx
\]

Which matches \( \frac{\partial f}{\partial y}(x, c) \).

(d) We expect the cross section to be a curve in \( \mathbb{R}^2 \) and to be given by an equation \( z = g(x) \). To find the \( y = x \) cross section, we can simply plug in \( x \) for \( y \) and have \( z = g(x) = f(x, x) \).

\[
g(x) = f(x, x) = x^3 + x^2
\]

3. Find the matrix of partial derivatives of the function

\[
F(x, y, z) = (ze^{x^2+2} + xy, \cos(x^3 y^2 z^4))
\]

Solution:
We see that \( F: \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) so we know that we will have a \( 2 \times 3 \) matrix, and that we will need to compute 6 partial derivatives.

To begin we will rewrite

\[
F(x, y, z) = (f_1(x, y, z), f_2(x, y, z))
\]

Where

\[
f_1(x, y, z) = z e^{x^2+y^2} + xy \\
f_2(x, y, z) = \cos(x^3y^2z^4)
\]

So we will compute all the partial derivatives independently

\[
\frac{\partial f_1}{\partial x} = y + 2xze^{x^2+y^2} \\
\frac{\partial f_1}{\partial y} = x + 2yz e^{x^2+y^2} \\
\frac{\partial f_1}{\partial z} = e^{x^2+y^2}
\]

\[
\frac{\partial f_2}{\partial x} = -3x^2y^2z^4 \sin(x^3y^2z^4) \\
\frac{\partial f_2}{\partial y} = -2x^3yz^4 \sin(x^3y^2z^4) \\
\frac{\partial f_2}{\partial z} = -4x^3y^2z^3 \sin(x^3y^2z^4)
\]

So we then put these partial derivatives into the matrix of partial derivatives as follows

\[
\begin{pmatrix}
y + 2xze^{x^2+y^2} & x + 2yz e^{x^2+y^2} & e^{x^2+y^2} \\
-3x^2y^2z^4 \sin(x^3y^2z^4) & -2x^3yz^4 \sin(x^3y^2z^4) & -4x^3y^2z^3 \sin(x^3y^2z^4)
\end{pmatrix}
\]

4. A function \( f(x, y) \) is harmonic if it satisfies the Laplace equation \( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \). Show that \( f(x, y) = x^3 - 3xy^2 \) is harmonic.

**Solution:** To show that this function is harmonic we start by taking the second order partial derivatives of \( f(x, y) \)
\[
\frac{\partial^2 f}{\partial x^2} = 6x \\
\frac{\partial^2 f}{\partial y^2} = -6x
\]

So for this specific \( f \) we know

\[
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 6x - 6x = 0
\]

Thus this function is harmonic.

5. The heat equation is: \( \frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2} \). Show that \( u(x, t) = e^{-k^2 t} \sin(x) \) is a solution of the heat equation.

**Solution:** We begin by computing the equations used in the heat equation:

\[
\frac{\partial u}{\partial t} = -e^{-k^2 t} \sin(x) \\
\frac{\partial^2 u}{\partial x^2} = -e^{-k^2 t} k^2 \sin(x)
\]

6. Let \( f(x, y) = x^2 + \frac{1}{2} y^2 - 2x \) Find a point on the graph \( z = f(x, y) \) where the tangent plane is horizontal.

**Solution:** We recall from in class, lab, textbooks, and folklore that we can write the normal vector at a point as

\[
\hat{n} = (\frac{-\partial f}{\partial x}, \frac{-\partial f}{\partial y}, 1)
\]

So for our purposes we will have

\[
\hat{n} = (-2x + 2, -y, 1)
\]

So all we need to find a point \((x, y)\) where \( \hat{n} = (0, 0, 1) \). The choice of \((1, 0)\) is one such point. \( f(1, 0) = -1 \) meaning that a point where the tangent plane is horizontal will be \((1, 0, -1)\)
Another good way to approach this part of the problem was to say “If the plane is horizontal then $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial x} = 0$. This gives the same solution.

7. Let $f(x, y) = x/y + y/x$. Using a linear approximation about the point $(1/2, 1/4)$, estimate the value of $f(.48, .3)$.

**Solution:** We begin by recalling that the tangent plane gives a linear approximation to $f(x, y)$ at a point, so we first compute the matrix of partial derivatives.

$$Df(x, y) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = \left( \frac{1}{y} - \frac{y}{x^2}, \frac{1}{x} - \frac{x}{y^2} \right)$$

Evaluating this at the point $(1/2, 1/4)$ we have

$$Df(1/2, 1/4) = \left( \frac{1}{1/4} - \frac{1/4}{(1/2)^2}, \frac{1}{1/2} - \frac{1/2}{(1/4)^2} \right) = (3, -6)$$

Now recall the form of the tangent plane $L(x, y)$ at the point $(a, b)$.

$$L(x, y) = f(a, b) + Df(a, b) \cdot (x - a, y - b)$$

and for the point $(1/2, 1/4)$ this becomes

$$L(x, y) = f(a, b) + Df(a, b) \cdot (x - a, y - b) = f(1/2, 1/4) + Df(1/2, 1/4) \cdot (x - 1/2, y - 1/4) = \frac{5}{2} + (3, -6) \cdot (x - 1/2, y - 1/4) = \frac{5}{2} + 3x - \frac{3}{2} - 6y + \frac{6}{4} = \frac{5}{2} + 3x - 6y$$

Now to estimate the value of $f(.48, .3)$ we compute
\[ L(.48, .3) = \frac{5}{2} + 3 \cdot (.48) - 6 \cdot (.3) = 2.14 \]

Using a computer we can compute the exact value to be 2.225, and we see that our estimate is fairly close.