

1 MacDonal constant term identites for type B_n

The MacDonal constant-term identity for type B_2 root systems says that the constant term $f_{0,0}$ in the identity

$$F(x, y) := (x, q/x, y, q/y, y/x, qx/y, xy, q/xy; q)_\infty = f_{0,0} \sum_{i,j} \text{exq} \left(3 \binom{j}{2} + 3 \binom{i}{2} + i \right) (-1)^{i+j} (x^{3i} y^{3j} - x^{3i+1} y^{3j} + x^{3i+2} - x^{3i+2} y^{3j+2})$$

equals $1/(q; q)_\infty^2$. I sometimes write “exq(x)” for “ q^x ”, analagous to $\exp(x)$ and e^x . To prove this identity, we’ll bang on the right side first and see what happens.

The right-hand side We’ll “sieve” the series to knock out the non-modulo-3 terms. Let ω be a primitive third root of unity ($e^{2\pi i/3}$ will do nicely). Consider $\frac{1}{3}(F(1, y) + F(\omega, y) + F(\omega^2, y))$. Ignore the q factors; the x and y terms are

$$\begin{aligned} & (y^{3j} - y^{3j} + y^{3j+1} - y^{3j+2}) \\ & (y^{3j} - \omega y^{3j} + \omega^2 y^{3j+1} - \omega^2 y^{3j+2}) \\ & (y^{3j} - \omega^2 y^{3j} + \omega y^{3j+1} - \omega y^{3j+2}) \end{aligned}$$

and since $1 + \omega + \omega^2 = 0$, the sum of the above terms is y^{3j} (don’t forget the $1/3$). Therefore the sieved sum can be written

$$\begin{aligned} f_{0,0} \sum_{i,j} \text{exq} \left(3 \binom{j}{2} + 3 \binom{i}{2} + i \right) (-1)^{i+j} y^{3j} \\ = f_{0,0} \left(\sum_i (-1)^i q^i q^{3 \binom{i}{2}} \right) \sum_j (-1)^j q^{3 \binom{j}{2}} y^{3j}. \end{aligned}$$

The sum over i can be rewritten using the Jacobi Triple Product identity: it sums to $(q^3, q, q^2; q^3)_\infty = (q; q)_\infty$. Now by looking at the $j = 0$ term from the other sum, we see that the constant term on the right-hand side is $(q; q)_\infty f_{0,0}$.

The left-hand side Now look at $\frac{1}{3}(F(1, y) + F(\omega, y) + F(\omega^2, y))$ and plug in the products (I’ll omit the one third, it just gets in the way). Remember that $1/\omega = \omega^2$.

$$\begin{aligned} & (1, q, y, q/y, y, q/y, y, q/y; q)_\infty + \\ & (\omega, q\omega^2, y, q/y, y\omega^2, q\omega/y, \omega y, q\omega^2 y; q)_\infty + (\omega^2, q\omega, y, q/y, y\omega, q\omega/y, \omega y, q\omega y; q)_\infty \end{aligned}$$

The first term is zero because it starts with $1 - 1q^0$. Now we can simplify a lot of things: for instance, the y , ωy , and $\omega^2 y$ terms turn into $(y^3; q^3)_\infty$ because $(1 - yq^i)(1 - \omega yq^i)(1 - \omega^2 yq^i) = (1 - y^3 q^{3i})$. The last two terms can be written

$$(\omega, q\omega^2; q)_\infty (y^3, q^3/y^3; q^3)_\infty + (\omega^2, q\omega; q)_\infty (y^3, q^3/y^3; q^3)_\infty.$$

Use Jacobi Triple Product on the q^3 factorials; they each equal

$$\frac{1}{(q^3; q^3)_\infty} \sum_m q^{3\binom{m}{2}} (-y^3)^m$$

and have constant terms equal to $1/(q^3; q^3)_\infty$. We can mess with the other factorials as follows:

$$(\omega, q\omega^2; q)_\infty = (1 - \omega)(q\omega, q\omega^2; q)_\infty = (1 - \omega) \frac{(q, q\omega, q\omega^2; q)_\infty}{(q; q)_\infty} = (1 - \omega) \frac{(q^3; q^3)_\infty}{(q; q)_\infty}.$$

Now the sieved product—which became those two terms—equals

$$(1 - \omega) \frac{1}{(q; q)_\infty} + (1 - \omega^2) \frac{1}{(q; q)_\infty} = \frac{3}{(q; q)_\infty}.$$

But we forgot about the one third, so toss that in, and our constant term equals $1/(q; q)_\infty$. Equate that with the constant term from the other side, and we see that

$$f_{0,0} = \frac{1}{(q; q)_\infty^2}.$$