University of Minnesota School of Mathematics
Written Preliminary Exam – Real Analysis
Fall 2002

[Typed by Dan Drake (drake@math.umn.edu). The math department has not reviewed or approved this copy.]

- $\mathbb{R}$ denotes the real line.
- Functions are real-valued unless otherwise stated.
- Integration is with respect to Lebesgue measure unless otherwise stated.

Part I

1. (10 points) Let $f(x)$ be a continuous function on $[0,1]$ and assume that for all $n = 1, 2, 3 \ldots,$

$$\int_0^1 f(t)t^n \, dt = 0.$$

Show that $f(x)$ vanishes identically; that is, $f(x) \equiv 0$.

2. (15 points) Denote by M the subset of continuous functions on $[0,1]$ with $f(0) = 0$ and Lipschitz constant $\text{Lip } f \leq 1$; that is,

$$|f(x) - f(y)| \leq |x - y| \text{ for all } x, y \in [0,1].$$

For $f, g \in M$ set

$$d_0(f,g) = \sup_x |f(x) - g(x)|;$$

$$d_1(f,g) = \sup_{x \neq 0} \frac{|f(x) - g(x)|}{|x|}.$$

Then $d_0(f,g)$ and $d_1(f,g)$ define metrics on M (you do not need to show this). Show that in both cases, M is complete. Also decide in both cases whether M is compact.

3. (15 points) For the subsets E_1, E_2, and E_3 defined below, decide whether E_i is bounded and whether it is compact in ℓ^2:

$$E_1 := \{x \in \ell^2 : |x_i| \leq \frac{1}{\sqrt{i}} \text{ for all } i\}$$

$$E_2 := \{x \in \ell^2 : \sum_{i=0}^{\infty} |x_i|^2 \leq 1\}$$

$$E_3 := \{x \in \ell^2 : |x_i| \leq \frac{1}{x_i} \text{ for all } i\}.$$

4. (10 points) Let X be a Hilbert space with scalar product $(\cdot\,,\cdot)$ and norm $\|\cdot\|$. Let K be a closed and convex subset of $z \notin K$ be some point outside of K.

   (a) Show that for all $x, y \in X$

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

(b) Let $d(z, K) := \inf\{\|z - x\| : x \in K\}$ denote the distance from z to K. Show that for a given z, every sequence $x_n \in K$ with

$$\lim_{n\to\infty} \|z - x_n\| = d(z, K)$$

is a Cauchy sequence.

(c) Show that there exists a unique $x_0 \in K$ such that $\|z - x_0\| = d(z, K)$, and for all $x \in K$,

$$(z - x_0, x - x_0) \leq 0.$$

Part II

1. (10 points) Suppose that $f(x)$ is continuous and of bounded variation on $[0, 1]$. Does it follow that
$$f(1) = \int_0^1 f'(x)\, dx + f(0) \text{ ?}$$
Give a proof or counterexample.

2. (10 points) Is $e^{x^2} \geq 1 + x^2$ everywhere? Give a proof or counterexample.

3. (10 points) Let $f_n(x)$ be a sequence of continuous functions on $(-\infty, \infty)$ which converges at every point. Determine whether the following is true or false: "There is a nonempty open interval I and a number $M < \infty$ so that $|f_n(x)| < M$ for every $x \in I$ and all n."

4. (10 points) Suppose $\varphi(x)$ is a positive increasing function with $x/\varphi(x) \to 0$ as $x \to \infty$. Also suppose $f_n(x)$, $n = 1, 2, 3, \ldots$ are measurable with
$$f_n(x) \to f(x) \text{ a.e. as } n \to \infty, \text{ and } \int_0^1 \varphi(|f_n(x)|)\, dx \leq 10$$
for all n. Show that
$$\int_0^1 |f_n(x) - f(x)|\, dx \to 0$$
as $n \to \infty$. (Hint: consider first the case where $f_n(x)$ are uniformly bounded.)

5. (10 points) Let $\mathcal{F}$ be a vector space consisting of a subset of the continuous functions on $[0, 1]$ and Φ a linear functional on $\mathcal{F}$ so that for $f \in \mathcal{F}$,
$$|\Phi(f)| \leq \int_0^1 |f(x)|\, dx.$$
Show that for some measurable $g(x)$ with $\int_0^1 |g(x)|\, dx < \infty$,
$$\Phi(f) = \int_0^1 f(x)g(x)\, dx \text{ for all } f \in \mathcal{F}.$$