A proof of the classic complex analysis prelim question: show that two annuli are conformally equivalent only if the ratios of their radii are the same. Shamelessly stolen from James Swenson's solutions (I think James shamelessly stole them from somewhere else...we've mastered Tom Lehrer's secret of success in mathematics!)

Without loss of generality say the annuli, call them A_1 and A_2, are centered at zero and have inner radius one (translations and contractions/dilations are conformal maps). Call their outer radii R_1 and R_2. Let f be a conformal map from A_1 to A_2. We'll show that $R_1 = R_2$.

Since f takes boundaries to boundaries, the image of the unit circle C is either itself or the circle of radius R_2; without loss of generality we can assume $f(C) = C$; otherwise, we could replace f with R_2/f which is also conformal because f is nonzero on its domain.

Now we pull a rabbit out of our hat. Let $u(z) := \log|f(z)| - \alpha \log|z|$, where $\alpha = \log R_2 / \log R_1$. This function is harmonic on A_1 because it's the difference of two harmonic functions; we know the two logarithms are harmonic because they're the real part of analytic functions. Note that u is identically zero for any z on the unit circle or the circle of radius R_1; by the Maximum Principle, that means u is identically zero on all of A_1. In particular, its derivative is zero.

Another rabbit. Find the derivative of $z^{-\alpha} f(z)$:

$$\begin{aligned}
\frac{d}{dz}[z^{-\alpha} f(z)] &= z^{-\alpha} f'(z) - \alpha z^{-\alpha-1} f(z) \\
&= z^{-\alpha} f(z) \left(\frac{f'(z)}{f(z)} - \frac{\alpha}{z} \right) \\
&= z^{-\alpha} f(z) \frac{du}{dz} \\
&= 0.
\end{aligned}$$

This means that $z^{-\alpha} f(z)$ is constant, so that $f(z) = cz^{\alpha}$. Since the unit circle is mapped to itself, we have $c = 1$, and since f must be 1-1 and onto, $\alpha = 1$; i.e., $R_1 = R_2$. If the two annuli are conformally equivalent, the ratios of their radii must be the same. $\square$

Dan Drake (drake@math.umn.edu)