

# Risk & Asset Allocation (Spring)

## Exercise for Week 4

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### 1 Efficient Frontier

With the objective  $\Psi_\alpha = \alpha' M$  for market vector  $M = P - p$  (profit objective) with initial price vector  $p$  (dollars per share) and portfolio allocation  $\alpha$  (shares), the efficient frontier is defined by

$$\alpha(m) = \arg \min_{\substack{\alpha \in \mathcal{C} \\ \alpha' E M \geq m}} \alpha' (\text{cov } M) \alpha$$

which is parameterized by  $m$ .

### 2 Quadratic Programming

A class of convex optimization problems that can be readily solved with polynomial-scale algorithms is termed quadratic programming,

$$\min_{Ax \leq b} x' H x$$

for  $H > 0$  (positive definite) and (not necessarily square) matrix  $A$  and corresponding vector  $b$ .

Note that this can be easily extended to include a linear term in the objective function (e.g.  $x \mapsto x + \frac{1}{2} H^{-1} c$ ) and equality constraints,

$$\begin{pmatrix} A'_1 \\ -A'_1 \end{pmatrix} x \leq \begin{pmatrix} b_1 \\ -b_1 \end{pmatrix}$$

### 3 Constraints

Since covariance is positive definite, as long as the constraints are affine, the efficient frontier problem is an example of quadratic programming.

Common constraints that can be expressed in this form, in addition to the minimum return constraint above, include the wealth constraint  $p'\alpha \leq w$  with initial wealth  $w$ , and the shorting prohibition  $\alpha \geq 0$ .

$$H = \text{cov } M \quad A = \begin{pmatrix} -\frac{w}{m} \text{E } M' \\ p' \\ -\text{diag } p \end{pmatrix} \quad b = w \begin{pmatrix} -1 \\ 1 \\ \text{diag } 0 \end{pmatrix}$$

Note that  $m$ , which is the lower bound on  $\text{E } \Psi_\alpha$ , is itself bounded above. The maximum value is the profit on a portfolio with all of the initial wealth invested in the asset with the highest return.

$$m_{\max} = w \max_i \frac{\text{E } M_i}{p_i}$$

## 4 Optimization

Note that solutions along the efficient frontier will probably not be “fully invested”. The uninvested portion,

$$w - p'\alpha(m)$$

represent an implicit cash allocation.

In this case, our investor’s objective is profit, so (assuming zero interest rates) cash does not contribute to this objective. In other circumstances, cash may contribute, so we need to be careful to include the contribution from the implicit cash allocation.

Assuming that the investor satisfaction  $\mathcal{S}(\alpha)$  is consistent with stochastic dominance, the optimal portfolio  $\alpha^*$  is on the efficient frontier, and we can use the two-step procedure to identify it.

$$\alpha^* = \arg \max_{0 \leq \beta \leq 1} \mathcal{S}(\alpha(\beta m_{\max}))$$

This is a (constrained) one-dimensional optimization. It can be a little intense, since you need to solve a quadratic programming problem at each step, and a little delicate, since  $\alpha(m)$  is not continuous; but it is generally surmountable using modern algorithms.