Risk & Asset Allocation (Spring)
Exercise for Week 2

John A. Dodson

January 25, 2012

Let us try out the two-step approach to determine the optimal portfolio under a single affine constraint with objective linear in the market vector and index of satisfaction equal to the Cornish-Fisher expansion of the 95% expected shortfall.

For an objective defined by $\Psi_\alpha = \alpha' M$ and an affine constraint defined by $d'\alpha = c$, the analytic solution to the optimal mean-variance portfolio satisfies

$$\alpha(\beta) = (1-\beta)\alpha_{MV} + \beta\alpha_{SR} \tag{1}$$

for $\beta > 0$, where

$$\alpha_{MV} = \frac{c\left(\mathsf{Cov} M\right)^{-1} d}{d'\left(\mathsf{Cov} M\right)^{-1} d}$$
$$\alpha_{SR} = \frac{c\left(\mathsf{Cov} M\right)^{-1} \mathsf{E} M}{d'\left(\mathsf{Cov} M\right)^{-1} \mathsf{E} M}$$

We need to determine the level of β that maximizes the index of satisfaction, which we will take to be

$$\mathcal{S}(\alpha) = \mathsf{E}\,\Psi_\alpha + \sqrt{\mathsf{Var}\,\Psi_\alpha}\left(\mathcal{I}\left[\phi\Phi^{-1}\right] + \tfrac{1}{6}\left(\mathcal{I}\left[\phi\left(\Phi^{-1}\right)^2\right] - 1\right)\mathsf{Skew}\,\Psi_\alpha\right)$$

based on the Cornish-Fisher expansion, where Φ is the CDF of a standard normal random variable and ϕ is the spectrum for $ES_{0.95}$.

Since we can assume that the skewness of M is negligible, the skewness of Ψ_α is also negligible. Furthermore, we can evaluate the integral in the expansion numerically.

$$\mathcal{I}\left[\phi\Phi^{-1}\right] = \int_0^{0.05} \frac{\sqrt{2}\operatorname{erf}^{-1}(2p-1)}{0.05}\,dp \approx -2.0627\cdots$$

Let us assign $z_{0.95} = 2.0627\cdots$, so the integral above is $-z_{0.95}$. The satisfaction is

$$\mathcal{S}(\alpha) = \alpha'\mathsf{E}M - z_{0.95}\sqrt{\alpha'\left(\mathsf{Cov}M\right)\alpha}$$

Substituting in (1), we get that the optimal value for β is

$$\begin{aligned}\beta^* = \arg\max_{\beta>0}\, &((1-\beta)\alpha_{MV} + \beta\alpha_{SR})'\,\mathsf{E}M \\ &- z_{0.95}\sqrt{((1-\beta)\alpha_{MV} + \beta\alpha_{SR})'\left(\mathsf{Cov}M\right)((1-\beta)\alpha_{MV} + \beta\alpha_{SR})}\end{aligned}$$

From manipulation of the first-order condition, recognizing that

$$\mathsf{Cov}\left(\Psi_{\alpha_{MV}}, \Psi_{\alpha_{SR}} - \Psi_{\alpha_{MV}}\right) = 0$$

we can determine that the solution is

$$\beta^* = \begin{cases} 0 & \gamma \leq 0 \\ \sqrt{\frac{\mathsf{Var}\,\Psi_{\alpha_{MV}}}{\mathsf{Var}\left(\Psi_{\alpha_{SR}} - \Psi_{\alpha_{MV}}\right)} \frac{1}{\left(\frac{z_{0.95}}{\gamma}\right)^2 - 1}} & 0 < \gamma < z_{0.95} \\ \infty & \gamma \geq z_{0.95} \end{cases}$$

where

$$\gamma = \frac{\mathsf{E}\left(\Psi_{\alpha_{SR}} - \Psi_{\alpha_{MV}}\right)}{\sqrt{\mathsf{Var}\left(\Psi_{\alpha_{SR}} - \Psi_{\alpha_{MV}}\right)}}$$

is the market price for risk.

In conclusion, the optimal portfolio is $\alpha(\beta^*)$.