Risk & Asset Allocation
Case Solution for Week 4

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Say X is the invariant random variable that will drive market prices $P_{T+\tau} \triangleq g(X; p_T)$ between today, time T, and the next decision date, time $T+\tau$ with an investment horizon of τ (all time measured in years) and say that we have used historical data to estimate the parameters of the characterization of the random variable $\tilde{X}$ under a different (usually shorter) horizon, $\tilde{\tau}$; e.g., from a timeseries sample $(p_T, p_{T-\tilde{\tau}}, p_{T-2\tilde{\tau}}, \ldots)$.

Furthermore, say our invariant mapping is additive in the sense that

$$g\left(\tilde{x}_1; g\left(\tilde{x}_2, p_{T-2\tilde{\tau}}\right)\right) = g\left(\tilde{x}_1 + \tilde{x}_2; p_{T-2\tilde{\tau}}\right)$$

which is true, for example, with the continuous version of total return (although the simple version of total return is often an adequate approximation).

Then projecting X from $\tilde{X}$ essentially involves the transformation

$$X \triangleq \overbrace{\tilde{X} + \tilde{X} + \cdots}^{\frac{\tau}{\tilde{\tau}} \text{ i.i.d. copies}}$$

We know from the properties of characteristic functions that as long as increments are independent and identically distributed,

$$\phi_X(\omega) = \phi_{\tilde{X}}(\omega)^{\frac{\tau}{\tilde{\tau}}}$$

If the first two moments exist, we also know that

$$\begin{aligned}
\mathrm{E}\, X &= -i \left.\frac{d\phi_X}{d\omega'}\right|_0 \\
&= -i \frac{\tau}{\tilde{\tau}} \phi_{\tilde{X}}^{\frac{\tau}{\tilde{\tau}}-1} \left.\frac{d\phi_{\tilde{X}}}{d\omega'}\right|_0 \\
&= \frac{\tau}{\tilde{\tau}} \mathrm{E}\, \tilde{X}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}\left(X X'\right) &= -\left.\frac{d^2\phi_X}{d\omega'\, d\omega}\right|_0 \\
&= -\frac{\tau}{\tilde{\tau}}\left(\frac{\tau}{\tilde{\tau}} - 1\right)\left(\phi_{\tilde{X}}\right)^{\frac{\tau}{\tilde{\tau}}-2} \frac{d\phi_{\tilde{X}}}{d\omega'} \frac{d\phi_{\tilde{X}}}{d\omega} - \frac{\tau}{\tilde{\tau}}\left(\phi_{\tilde{X}}\right)^{\frac{\tau}{\tilde{\tau}}-1} \left.\frac{d^2\phi_{\tilde{X}}}{d\omega'\, d\omega}\right|_0 \\
&= \frac{\tau}{\tilde{\tau}}\left(\frac{\tau}{\tilde{\tau}} - 1\right) \mathrm{E}\, \tilde{X}\, \mathrm{E}\, \tilde{X}' + \frac{\tau}{\tilde{\tau}} \mathrm{E}\left(\tilde{X}\tilde{X}'\right) \\
&= \mathrm{E}\, X\, \mathrm{E}\, X' + \frac{\tau}{\tilde{\tau}}\left(\mathrm{E}\left(\tilde{X}\tilde{X}'\right) - \mathrm{E}\, \tilde{X}\, \mathrm{E}\, \tilde{X}'\right)
\end{aligned}$$

Since the covariance is defined as

$$\operatorname{cov} \tilde{X} = \mathrm{E}\left(X X'\right) - \mathrm{E}\, X\, \mathrm{E}\, X'$$

we have that

$$\operatorname{cov} X = \tfrac{\tau}{\tilde{\tau}} \operatorname{cov} \tilde{X}$$

Furthermore, since

$$\operatorname{std} \tilde{X} = \operatorname{diag} \sqrt{\operatorname{diag} \operatorname{diag} \operatorname{cov} \tilde{X}}$$

we have the "square-root rule" for time-scaling market invariants.

$$\operatorname{std} X = \sqrt{\tfrac{\tau}{\tilde{\tau}}} \operatorname{std} \tilde{X}$$

This is valid regardless of the distribution of $\tilde{X}$ (as long as it has two moments).

Note that in general X will not belong to the same family of random variables as $\tilde{X}$.