

Ch. 7 Intensity Models

Schönbucher, Philipp J. (2003)
Credit derivatives pricing models, Wiley Finance

John Dodson
finmath practitioner seminar

July 11, 2007

Introduction

Short Rate

Tractable
Gaussian model
CIR modelTree
Finite difference

Forward rates

HJM framework
Monte Carlo

Conclusion

Introduction

- ▶ chapter 7 is a continuation of chapter 3
- ▶ but we will drop Assumption 3.2, independence
- ▶ ...and specify (risk-neutral) stochastic dynamics
- ▶ since the building block securities can be expressed as

$$B(t, T) = \mathbb{E}^Q \left[e^{-\int_t^T r(s) ds} \middle| \mathcal{F}_t \right]$$

$$\bar{B}(t, T) = \mathbb{E}^Q \left[e^{-\int_t^T r(s) + \lambda(s) ds} \middle| \mathcal{F}_t \right]$$

$$e(t, T) = \mathbb{E}^Q \left[\lambda(T) \cdot e^{-\int_t^T r(s) + \lambda(s) ds} \middle| \mathcal{F}_t \right]$$

we will draw inspiration from interest rate models

- ▶ **N.B.:** we assume $\exists \lambda$, a (risk-neutral) default intensity

Introduction

Short Rate

Tractable
Gaussian model
CIR modelTree
Finite difference

Forward rates

HJM framework
Monte Carlo

Conclusion

Introduction

- ▶ there are two main classes of interest rate models
 - ▶ short rate
 - ▶ forward rate
- ▶ short rate models can be tractable
- ▶ or at least amenable to efficient numerical techniques
 - ▶ tree methods
 - ▶ finite difference methods
- ▶ forward rate models have more flexible dynamics
- ▶ but are generally difficult to compute
 - ▶ simulation methods

Ch. 7 Intensity Models

John Dodson
finmath
practitioner
seminar

Introduction

Short Rate

Tractable
Gaussian model
CIR model
Tree
Finite difference

Forward rates

HJM framework
Monte Carlo

Conclusion

Short rate models

short rate models start with a process description of the short-term interest rate evolving under the risk-neutral measure

- ▶ tractable models trade flexibility for simplicity & intuition
- ▶ each has certain flaws
 - ▶ Gaussian: potentially negative probabilities
 - ▶ Affine: strictly positive rate-spread correlation
- ▶ ...but they can be considered to be locally valid
- ▶ numerical models are much more flexible
- ▶ short-rate models are inherently Markovian, so efficient to evaluate

Ch. 7 Intensity Models

John Dodson
finmath
practitioner
seminar

Introduction

Short Rate

Tractable
Gaussian model
CIR model
Tree
Finite difference

Forward rates

HJM framework
Monte Carlo

Conclusion

Gaussian model

simplest stochastic intensity model

- ▶ the Gaussian model in §7.1 is based on the Vasicek model for interest rates
- ▶ the key dynamical assumption is that innovations in rates are normal
- ▶ Lemma 7.1 is the main analytical workhorse

$$dx(t) = (\kappa(t) - \alpha \cdot x(t)) dt + \sigma(t) dW(t)$$
$$\implies E \left[e^{-\int_t^T x(s) ds} \middle| \mathcal{F}_t \right] = e^{\mathcal{A}(t,T) - \mathcal{B}(t,T) \cdot x(t)}$$

where \mathcal{A} and \mathcal{B} are defined in (7.10) and (7.9)

- ▶ Proposition 7.2 uses these to derive solutions for the building blocks where $r(t)$ and $\lambda(t)$ are combinations of two Gaussian processes

Ch. 7 Intensity Models

John Dodson
finmath
practitioner
seminar

Introduction

Short Rate

Tractable
Gaussian model
CIR model

Tree
Finite difference

Forward rates

HJM framework
Monte Carlo

Conclusion

Multifactor gaussian model

general framework for correlation

- ▶ the factors are latent in the multifactor gaussian model
- ▶ instead we start with the risk-neutral process definitions of the zerobond values

$$\frac{dB(t, T)}{B(t, T)} = r(t) dt + \vec{a}(t, T)' d\vec{W}(t)$$
$$\frac{d\bar{B}(t, T)}{\bar{B}(t-, T)} = (r(t) + \lambda(t)) dt + \vec{a}(t, T)' d\vec{W}(t) - dN(t)$$

where the vector form of the volatility allows for a completely general description of the correlation of innovations between the curves and amongst points on the curves over time

Ch. 7 Intensity Models

John Dodson
finmath
practitioner
seminar

Introduction

Short Rate

Tractable
Gaussian model
CIR model

Tree
Finite difference

Forward rates

HJM framework
Monte Carlo

Conclusion

Multifactor gaussian model

general framework for correlation

- ▶ in this setting, where B and \bar{B} are taken as given, the remaining building block security is

$$e(t, T) = \bar{B}(t, T).$$

$$\left[h(t, T) - \int_t^T \vec{a}(s, T)' \frac{\partial}{\partial T} (\vec{a}(s, T) - \bar{a}(s, T)) ds \right]$$

where $h = \bar{f} - f$ is the forward hazard rate

- ▶ the last term reflects the correlation between r and λ
- ▶ **N.B.:** I believe there are typos in (7.23) and (7.24)

Ch. 7 Intensity Models

John Dodson
finmath
practitioner
seminar

Introduction

Short Rate

Tractable
Gaussian model
CIR model

Tree
Finite difference

Forward rates

HJM framework
Monte Carlo

Conclusion

Cox-Ingersoll-Ross model

- ▶ the affine model in §7.2-3 is based on the Cox-Ingersoll-Ross model for interest rates
- ▶ the key dynamical assumption is that innovations in rates are ANC: affine combinations of non-central chi-squared random variables
- ▶ say there are n independent factors x_i , where

$$dx_i(t) = (\alpha_i - \beta_i \cdot x_i(t)) dt + \sigma_i \cdot \sqrt{x_i(t)} dW_i(t)$$

with positive parameters and $\alpha_i > \sigma_i^2/2$ to insure that $x_i(t) > 0$ a.s.

- ▶ analogously to the gaussian model, the main analytical workhorse for this model is

$$\mathbb{E} \left[e^{-\int_t^T \sum_i c_i \cdot x_i(s) ds} \middle| \mathcal{F}_t \right] = e^{\sum_i \log H_{1i}(T-t, c_i) - H_{2i}(T-t, c_i) \cdot c_i \cdot x_i(t)}$$

where H_1 and H_2 are defined in (7.29) and (7.30)

Ch. 7 Intensity Models

John Dodson
finmath
practitioner
seminar

Introduction

Short Rate

Tractable
Gaussian model
CIR model

Tree
Finite difference

Forward rates

HJM framework
Monte Carlo

Conclusion

Cox-Ingersoll-Ross model

Ch. 7 Intensity Models

John Dodson
finmath
practitioner
seminar

- ▶ define the short rate and default intensity by

$$r(t) = \sum_{i=1}^n w_i \cdot x_i(t)$$
$$\lambda(t) = \sum_{i=1}^n \bar{w}_i \cdot x_i(t)$$

with w_i and \bar{w}_i non-negative

- ▶ this guarantees that rates are non-negative
- ▶ ...but also limits the model to positive correlations
- ▶ there are a total of $5n + 2$ parameters to fit to calibrate the model

Introduction

Short Rate

Tractable
Gaussian model
CIR model

Tree
Finite difference

Forward rates

HJM framework
Monte Carlo

Conclusion

Cox-Ingersoll-Ross model

Ch. 7 Intensity Models

John Dodson
finmath
practitioner
seminar

with the workhorse and the result of Proposition 7.8, we can write down the values of the building block securities

$$B(t, T) = e^{\sum_{i=1}^n \log H_{1i}(T-t, w_i) - H_{2i}(T-t, w_i) \cdot w_i \cdot x_i(t)}$$

$$\bar{B}(t, T) = e^{\sum_{i=1}^n \log H_{1i}(T-t, w_i + \bar{w}_i) - H_{2i}(T-t, w_i + \bar{w}_i) \cdot (w_i + \bar{w}_i) \cdot x_i(t)}$$

and

$$e(t, T) = \bar{B}(t, T) \cdot$$

$$\sum_{i=1}^N \bar{w}_i \cdot (w_i + \bar{w}_i) \cdot \left(\alpha_i + x_i(t) \cdot \frac{\partial}{\partial T} \right) H_{2i}(T - t, w_i + \bar{w}_i)$$

Introduction

Short Rate

Tractable
Gaussian model
CIR model

Tree
Finite difference

Forward rates

HJM framework
Monte Carlo

Conclusion

Tree model

- ▶ the tree model in §7.4 is based on the Hull-White trinomial model for interest rates
 - ▶ which in turn is based on the gaussian model
- ▶ there are a total of ten successors to each node
 - ▶ although most of these are successors are shared—the tree is re-combining
- ▶ in order to prevent rates from going negative, the excursion is artificially limited by having the tree fold back on itself
- ▶ the risk-neutral branching probabilities are calibrated according to
 - ▶ (7.78-86) to fit the moments of the dynamics
 - ▶ tables 7.1-3 to incorporate the correlation
- ▶ note that tree models are essentially explicit schemes that implement the PDE in the next section

Ch. 7 Intensity Models

John Dodson
finmath
practitioner
seminar

Introduction

Short Rate

Tractable
Gaussian model
CIR model

Tree
Finite difference

Forward rates

HJM framework
Monte Carlo

Conclusion

Tree model

Hull-White gaussian model

in the Hull-White short rate model, the mean reversion level is allowed to vary deterministically in order to facilitate calibration

$$dr(t) = [k(t) - a \cdot r(t)] dt + \sigma dW(t)$$

the solution to this is

$$r(t) = r^*(t) + \alpha(t)$$

where the auxiliary process, r^* , is defined by $r^*(0) = 0$ and

$$dr^* = -a \cdot r^* dt + \sigma dW$$

and the deterministic offset

$$\alpha(t) = r(0) \cdot e^{-a \cdot t} + \int_0^t e^{-a \cdot (t-s)} \cdot k(s) ds$$

Ch. 7 Intensity Models

John Dodson
finmath
practitioner
seminar

Introduction

Short Rate

Tractable
Gaussian model
CIR model

Tree
Finite difference

Forward rates

HJM framework
Monte Carlo

Conclusion

Tree model

Hull-White trinomial tree model

- ▶ the model has six parameters: a , σ , \bar{a} , $\bar{\sigma}$, ρ , and π
- ▶ once these are specified,
 - ▶ trees in r^* and λ^* can be constructed; and
 - ▶ $\alpha(\{0 : T\})$ and $\bar{\alpha}(\{0 : T\})$ can be fit to $B(0, \{0 : T\})$ and $\bar{B}(0, \{0 : T\})$ by forward induction, starting with the risk-free curve
- ▶ the risk-neutral default branching probability is $p = 1 - e^{-\lambda \cdot \Delta t}$ from each non-default node
 - ▶ the recovery in the default node is π
 - ▶ should it be needed, the local expectation of the default time is $\tau^e = t + \frac{1}{\lambda} \cdot \left(1 - \frac{\lambda \cdot \Delta t}{e^{\lambda \cdot \Delta t} - 1}\right)$
- ▶ once calibrated, the tree can be used to price defaultable securities by backwards induction
- ▶ early exercise can be modeled by evaluating the early exercise option at each step

Ch. 7 Intensity Models

John Dodson
finmath
practitioner
seminar

Introduction

Short Rate

Tractable
Gaussian model
CIR model

Tree
Finite difference

Forward rates

HJM framework
Monte Carlo

Conclusion

Partial differential equation

- ▶ to derive the partial differential equation for a defaultable security in the intensity setting, start by specifying the stochastic processes for the rates

$$dr = \mu_r dt + \sigma_r dW_1$$

$$d\lambda = \mu_\lambda dt + \sigma_\lambda \cdot \left(\rho dW_1 + \sqrt{1 - \rho^2} dW_2 \right)$$

- ▶ also, let the compensator measure for the marked point process be

$$\nu(dt, d\pi) = \lambda(t)dt K(d\pi)$$

- ▶ and let the cashflow densities be

$$\begin{array}{ll} \tilde{f}(t, r, \lambda) & \text{prior to default} \\ g(t, r, \lambda, \pi) & \text{at default} \end{array}$$

Ch. 7 Intensity Models

John Dodson
finmath
practitioner
seminar

Introduction

Short Rate

Tractable
Gaussian model
CIR model

Tree
Finite difference

Forward rates

HJM framework
Monte Carlo

Conclusion

Partial differential equation

- ▶ define the value of the security to be

$$V(t) = v(t, r(t), \lambda(t)) \quad \text{for } t < \tau$$

- ▶ applying Itô's Lemma and the fundamental pricing rule, we get that v must satisfy

$$\partial_t v + \mathcal{L}v - (r + \lambda) \cdot v = -g^e \cdot \lambda - \tilde{f}$$

where the diffusion's linear operator is

$$\mathcal{L} = \mu_r \cdot \partial_r + \frac{1}{2} \cdot \sigma_r^2 \cdot \partial_{rr} + \mu_\lambda \cdot \partial_\lambda + \frac{1}{2} \cdot \sigma_\lambda^2 \cdot \partial_{\lambda\lambda} + \rho \cdot \sigma_r \cdot \sigma_\lambda \cdot \partial_{\lambda r}$$

and the locally (risk-neutral) expected default payoff is

$$g^e(t, r, \lambda) = \int_0^1 g(t, r, \lambda, \pi) K(d\pi)$$

Ch. 7 Intensity Models

John Dodson
finmath
practitioner
seminar

Introduction

Short Rate

Tractable
Gaussian model
CIR model

Tree
Finite difference

Forward rates

HJM framework
Monte Carlo

Conclusion

Partial differential equation

In these terms, a valuation formula v for any defaultable claim can be found given:

1. a final condition defined in terms of the no-default payoff (*may not be so simple*)

$$v(T, r, \lambda) = F(r, \lambda)$$

2. boundary conditions on v at the extremes of r and λ

- ▶ the problem is comparable to that for a default-free interest-sensitive claim
- ▶ in particular, adding a stochastic intensity and recovery effectively transforms:

$$\text{discount rate } r \rightarrow r + \lambda$$

$$\text{dividend rate } \tilde{f} \rightarrow \tilde{f} + g^e$$

Ch. 7 Intensity Models

John Dodson
finmath
practitioner
seminar

Introduction

Short Rate

Tractable
Gaussian model
CIR model

Tree
Finite difference

Forward rates

HJM framework
Monte Carlo

Conclusion

Forward rates framework

- ▶ the whole curve model in §7.6 is based on Heath-Jarrow-Morton
 - ▶ all short rate models are consistent with HJM
 - ▶ HJM with deterministic volatility is equivalent to the gaussian model
- ▶ since the dynamics are essentially unlimited, need to specify the no-arbitrage drift restriction
- ▶ model is non-Markovian, so valuation is in a simulation setting
- ▶ an implementation would proceed in the following steps
 1. specify initial risk-free term structure $f(0, \{0 : T\})$
 2. ...zero-recovery spread term structure $h(0, \{0 : T\})$
 3. ...forward rate volatilities and correlations
 4. calculate drifts
 5. simulate paths in $\beta, B, \bar{B}, e, I,$ and π
 6. evaluate pathwise security values
 7. average

Ch. 7 Intensity Models

John Dodson
finmath
practitioner
seminar

Introduction

Short Rate

Tractable
Gaussian model
CIR model

Tree
Finite difference

Forward rates

HJM framework
Monte Carlo

Conclusion

Heath-Jarrow-Morton

we start by taking as given,

1. the rates and spreads for $T \geq 0$

$$f(0, T) \quad \text{and} \quad h(0, T)$$

2. and the $i = 1, \dots, n$ co-volatilities for $t \geq 0$ and $T \geq t$

$$\sigma_i(t, T) \quad \text{and} \quad \sigma_i^h(t, T) \quad \forall t < \tau$$

- ▶ and define the risk-neutral processes for $0 \leq t \leq T$,

$$df(t, T) = \alpha(t, T) dt + \sum_{i=1}^n \sigma_i(t, T) dW_i$$

$$dh(t, T) = \alpha^h(t, T) dt + \sum_{i=1}^n \sigma_i^h(t, T) dW_i \quad \forall t < \tau$$

Ch. 7 Intensity Models

John Dodson
finmath
practitioner
seminar

Introduction

Short Rate

Tractable
Gaussian model
CIR model

Tree
Finite difference

Forward rates

HJM framework
Monte Carlo

Conclusion

Heath-Jarrow-Morton

no-arbitrage imposes the drift restrictions

$$\alpha(t, T) = \sum_{i=1}^n \sigma_i(t, T) \cdot \int_t^T \sigma_i(t, T') dT'$$

and

$$\alpha^h(t, T) = \sum_{i=1}^n \left[\sigma_i^h(t, T) \cdot \int_t^T \sigma_i(t, T') dT' + \left(\sigma_i(t, T) + \sigma_i^h(t, T) \right) \cdot \int_t^T \sigma_i^h(t, T') dT' \right]$$

also, Proposition 7.11 verifies that

$$h(t, t) = \lambda(t)$$

which can be used to model the default indicator process

Ch. 7 Intensity Models

John Dodson
finmath
practitioner
seminar

Introduction

Short Rate

Tractable
Gaussian model
CIR model

Tree
Finite difference

Forward rates

HJM framework
Monte Carlo

Conclusion

Monte Carlo

to value a defaultable security under an intensity model, we need to evaluate the pathwise integral in (7.143),

$$\begin{aligned} & \mathbb{E}^{\mathbb{Q}} [\beta(0, T) \cdot I(T) \cdot F(T) | \mathcal{F}_0] \\ & + \mathbb{E}^{\mathbb{Q}} [\beta(0, \tau) \cdot N(T) \cdot g(\tau, \pi) | \mathcal{F}_0] \\ & \mathbb{E}^{\mathbb{Q}} \left[\int_0^T \beta(0, t) \cdot I(t) \cdot \tilde{f}(t) dt \middle| \mathcal{F}_0 \right] \end{aligned}$$

where

- ▶ F is the final payoff at T if $T < \tau$
 - ▶ may depend on $f(T, t)$ and $h(T, t)$ for $T \leq t < \bar{T}$
- ▶ $g(t, \pi)$ is the payoff at default for recovery π if $t = \tau$
- ▶ $\tilde{f}(t)$ is the dividend rate for $t < \tau \wedge T$

and β, τ, N, I , and π all depend on the path ω

Ch. 7 Intensity Models

John Dodson
finmath
practitioner
seminar

Introduction

Short Rate

Tractable
Gaussian model
CIR model

Tree
Finite difference

Forward rates

HJM framework
Monte Carlo

Conclusion

Monte Carlo

scheme for rates

the HJM framework is well-suited for an Euler scheme to simulate pathwise forward rates

- ▶ time from $t = 0$ to $\bar{T} \geq T$ is discretized
- ▶ forward rates are updated according to

$$\begin{aligned}f(t_m, t_l) &= f(t_{m-1}, t_l) + \alpha(t_m, t_l) \cdot (t_m - t_{m-1}) \\ &\quad + \sum_{i=1}^n \sigma_i(t_m, t_l) \cdot \epsilon_{i,m} \cdot \sqrt{t_m - t_{m-1}} \\ h(t_m, t_l) &= h(t_{m-1}, t_l) + \alpha^h(t_m, t_l) \cdot (t_m - t_{m-1}) \\ &\quad + \sum_{i=1}^n \sigma_i^h(t_m, t_l) \cdot \epsilon_{i,m} \cdot \sqrt{t_m - t_{m-1}}\end{aligned}$$

for $l \geq m$ and $\epsilon_{i,m}$ standard i.i.d. variates

Ch. 7 Intensity Models

John Dodson
finmath
practitioner
seminar

Introduction

Short Rate

Tractable
Gaussian model
CIR model
Tree
Finite difference

Forward rates

HJM framework
Monte Carlo

Conclusion

Monte Carlo

schemes for default

by way of illustration, the book describes three schemes of increasing efficiency for simulating default

- ▶ fixed time grid
 - ▶ draw a uniform variate at each step in each path
 - ▶ default immediately if
$$-\log U_m < h(t_m, t_m) \cdot (t_m - t_{m-1})$$
- ▶ direct simulation of default time
 - ▶ draw a single uniform variate for each path
 - ▶ set τ to be the lowest t_M with
$$-\log U < \sum_{m=1}^M h(t_m, t_m) \cdot (t_m - t_{m-1})$$
- ▶ simulation with branching to default
 - ▶ do not simulate default at all;
apply iterated expectations instead

the author claims the latter method converges much quicker

Ch. 7 Intensity Models

John Dodson
finmath
practitioner
seminar

Introduction

Short Rate

Tractable
Gaussian model
CIR model
Tree
Finite difference

Forward rates

HJM framework
Monte Carlo

Conclusion

Monte Carlo

simulation with branching to default

analogously to Figure 3.2, calculate the (risk-neutral) expected defaultable value at each step in the tree

for each path,

- ▶ simulate all rates f and h out to \bar{T}
- ▶ set $\beta = \gamma = 1$ and $v = 0$ at $t = 0$
- ▶ value the security along the path from $t = 0$ to T
 1. update the discount factor: $\beta \leftarrow \beta \cdot e^{-f(t,t) \cdot \Delta t}$
 2. calculate the default probability: $p = 1 - e^{-h(t,t) \cdot \Delta t}$
 3. update the survival probability: $\gamma \leftarrow \gamma \cdot (1 - p)$
 4. accumulate the value:
$$v \leftarrow v + \beta \cdot \gamma \cdot \left(\tilde{f}(t) \cdot \Delta t \cdot (1 - p) + g^e(t) \cdot p \right)$$
 - ▶ where $g^e(t) = E^Q [g(t, \pi) | \mathcal{F}_t]$
 - ▶ **N.B.:** include the payoff $F(T)$ in the last step

...then average the values, v , over the paths

Ch. 7 Intensity Models

John Dodson
finmath
practitioner
seminar

Introduction

Short Rate

Tractable
Gaussian model
CIR model
Tree
Finite difference

Forward rates

HJM framework
Monte Carlo

Conclusion

Conclusion

- ▶ questions?
- ▶ remaining chapters

ch. 5	Cox process	Chris Bemis
ch. 6	recovery models	
ch. 8	transition models	John Baxter
ch. 9	structural model	Bill Barr
ch. 10	correlation models	Carlos Tolmasky
- ▶ papers
 - ▶ Duffie-Lando (2001), Term structures of credit spreads with incomplete accounting information
 - ▶ Andersen-Sidenius-Basu (2003), All your hedges in one basket
 - ▶ Carr-Flesaker (2006), Robust replication of default contingent claims
 - ▶ Errais-Giesecke-Goldberg (2007), Pricing credit from the top down with affine point processes

Ch. 7 Intensity Models

John Dodson
finmath
practitioner
seminar

Introduction

Short Rate

Tractable
Gaussian model
CIR model
Tree
Finite difference

Forward rates

HJM framework
Monte Carlo

Conclusion

Calendar

seven Wednesdays left before Fall term starts
on September 4 (Labour Day is September 3)

- ▶ July 18
- ▶ July 25
- ▶ August 1
- ▶ August 8
- ▶ August 15
- ▶ August 22
- ▶ August 29 (State Fair!)

Ch. 7 Intensity
Models

John Dodson
finmath
practitioner
seminar

Introduction

Short Rate

Tractable
Gaussian model
CIR model
Tree
Finite difference

Forward rates

HJM framework
Monte Carlo

Conclusion