Week 5: Operads and iterated loop spaces

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Operads

For a thorough introduction to the basic language regarding operads, take a look at Peter May's "Definitions: operads, algebras and modules" at

http://math.uchicago.edu/~may/PAPERS/handout.pdf

In our language, the operad we called Assoc is the operad $\mathcal{M}$ of Example 6, and the operad Com is $\mathcal{N}$ of Example 7.

The little disks operads

For each positive integer k, a ***TD-map*** $f : D^k \to D^k$ is the composite of a translation and a (positive) dilation[1]:

$$f(x) = ax + b,\ a \in \mathbb{R}_{>0},\ \text{and } b \in \mathbb{R}^k.$$

Definition 1. The k-dimensional ***little disks operads*** $\mathcal{C}_k$ is defined by the spaces

$$\mathcal{C}_k(n) := \{(f_1, \ldots, f_n) \mid f_i \text{ are TD-maps, and } f_i \operatorname{Int}(D^k) \cap f_j \operatorname{Int}(D^k) = \emptyset \text{ if } i \neq j\}.$$

We may topologize $\mathcal{C}_k(n)$ as a subspace of $(\mathbb{R}_{>0} \times \mathbb{R}^k)^{\times n}$ via the map which carries $(f_i)_{i=1}^n$ to $(a_i, b_i)_{i=1}^n$ The action of S_n on $\mathcal{C}_k(n)$ is by permutation of the f_i.

To define the operad composition γ, we note that an element $(f_1, \ldots, f_n) \in \mathcal{C}_k(n)$ defines a function $F : \coprod_n D^k \to D^k$ whose value on the i^{th} copy of D^k is given by f_i. By assumption, this map is injective on the interior of the domain. For elements $G^{(i)} = (g_1^{(i)}, \ldots, g_{j_i}^{(i)}) \in \mathcal{C}_k(j_i)$, we may define

$$\gamma(F; G^{(1)}, \ldots, G^{(n)}) = F \circ (G^{(1)} \sqcup \cdots \sqcup G^{(n)}) : \coprod_{\sum j_i} D^k \to D^k.$$

[1] If we additionally allow rotations, we obtain the notion of a ***TDR-map***; the associated operad is called the *framed little disks operad.*

The unit of the operad is the identity map $\mathrm{id}_{D^k} \in \mathcal{C}_k(1)$. We leave it to the reader to verify that this does indeed define an operad of topological spaces.

Consider the map $p : \mathcal{C}_k(n) \to \mathrm{PConf}_n(D^k)$ which carries a collection of little k-disks to their centers; that is, $p[(f_i)_{i=1}^n] = (b_i)_{i=1}^n$.

Proposition 2. *The map* $p : \mathcal{C}_k(n) \to \mathrm{PConf}_n(D^k)$ *is a homotopy equivalence.*

Proof. For a configuration $\underline{x} \in \mathrm{PConf}_n(D^k)$, define

$$d(\underline{x}) = \inf\{\frac{1}{2}|x_i - x_j|, i \neq j; |x_i - z|, z \in \partial D^k\}$$

Then $d : \mathrm{PConf}_n(D^k) \to \mathbb{R}_{>0}$ is a continuous function. Define $s : \mathrm{PConf}_n(D^k) \to \mathcal{C}_k(n)$ by $s(\underline{x}) = (f_1, \ldots, f_n)$, where

$$f_i(z) = d(\underline{x})z + x_i.$$

It is immediate that $p \circ s = \mathrm{id}$. In the other direction, if $f_i(z) = a_i z + b_i$, the i^{th} component of $s(p(f_1, \ldots, f_n))$ is the map $z \mapsto d(\underline{b})z + b_i$. A homotopy $s \circ p \simeq \mathrm{id}$ on $\mathcal{C}_k(n)$ is given by the straightline homotopy from a_i to $d(\underline{b})$.

□

Define a map $q : \mathcal{C}_1(n) \to \mathrm{Assoc}(n)$ which carries a configuration $(f_1, \ldots, f_n)$ of little disks in D^1 to the unique permutation $\sigma \in S_n$ with the property that $b_{\sigma(1)} < \cdots < b_{\sigma(n)}$. It is not hard to show that q is a map of operads.

Corollary 3. *The map* $q : \mathcal{C}_1 \to \mathrm{Assoc}$ *is a homotopy equivalence of operads.*

Proof. We have shown that $\mathcal{C}_1(n) \simeq \mathrm{PConf}_n(D^1)$. In the previous lecture, we showed[2] that $\mathrm{PConf}_n(D^1)$ is homeomorphic to $\mathrm{Int}(\Delta^n) \times S_n$. The projection map from this space onto $S_n = \mathrm{Assoc}(n)$ is precisely q, and is clearly a homotopy equivalence.

□

It is worth mentioning that while q has a homotopy inverse in the category of (sequences of) spaces, there is no homotopy inverse which is a map of operads. If A is an algebra for Assoc (that is, a strictly associative, unital H-space), then the composite

$$q : \mathcal{C}_1 \to \mathrm{Assoc} \to \mathrm{End}_A$$

makes A an algebra for $\mathcal{C}_1$. On the other had, the non-existence of an operadic section of q ensures that there are $\mathcal{C}_1$-algebras that are not strictly associative H-spaces. However, we will see in the future that every $\mathcal{C}_1$-algebra is homotopy equivalent to an Assoc-algebra.

[2]Or rather, we showed the corresponding statement for the unordered configuration space; the same proof holds here.

The operadic point of view underscores a general philosophy in "homotopy coherent mathematics:" equalities between operations often must be replaced by homotopies. However, homotopies themselves are usually too unstructured a notion to be of much use. If, however, those homotopies are encoded as paths in a structured object like an operad, a more useful[3] notion arises.

Example 4. The projection map $p : \mathcal{C}_k \to \mathrm{Com}$ which collapses everything in $\mathcal{C}_k(n)$ to a point is trivially a map of operads. For $k > 1$, it is an isomorphism in π_0. Consequently, every $\mathcal{C}_k$-algebra has a homotopy commutative multiplication. This may be seen explicitly as follows for $k = 2$: an arbitrary element $\mu \in \mathcal{C}_2(2)$ defines a multiplication on a $\mathcal{C} - 2$-algebra A. A Dehn twist around the midpoint of the centers of the little disks in μ gives a path from μ to $\mu \circ \sigma$, where $\sigma = (12)$ is the permutation swapping the two inputs. This yields a homotopy between $a \star b = \mu(a, b)$ and $b \star a = \mu \circ \sigma(a, b)$.

The recognition theorem

The recognition theorem asserts that *connected*[4] $\mathcal{C}_k$-algebras and k-fold loop spaces are essentially the same thing. Half of this is straightforward.

Proposition 5. *For any based space* $(X, *)$ *and integer* $k > 0$, $\Omega^k X = \mathrm{Map}((D^k, \partial), (X, *))$ *is a* $\mathcal{C}_k$*-algebra.*

Proof. Let $g_1, \dots, g_n \in \Omega^k X$ and $f = (f_1, \dots, f_n) \in \mathcal{C}_k(n)$ (so that the f_i are TD-maps). Define the action of $\mathcal{C}_k$ on $\Omega^k X$ as follows: for $d \in D^k$, the value of $\theta(f; (g_1, \dots, g_n))$ on d is

$$\theta(f; (g_1, \dots, g_n))(d) := \begin{cases} *, & d \notin \mathrm{im}\,(f_i) \text{ for any } i. \\ g_i f_i^{-1}(d), & d \in \mathrm{im}\,(f_i). \end{cases}$$

An enthralling computation verifies that this does indeed satisfy the axioms of an operad action.

□

This has a sort of converse, which is much less straightforward, and will require some substantial work:

Theorem 6 (Recognition, [May72]). *If* Y *is a connected* $\mathcal{C}_k$*-algebra, there exists a space* X *and a weak equivalence of* $\mathcal{C}_k$*-algebras* $\Omega^k Y \simeq X$.

[3]For instance, a homotopy associative, homotopy unital H-space does not have a well-structured category of modules, whereas an algebra over $\mathcal{C}_1$ does.

[4]In the disconnected case, look forward to the group completion theorem in the near future.

Free algebras and the approximation theorem

Definition 7. Let $(Z, *)$ be a pointed topological space and $k > 0$ an integer. The free $\mathcal{C}_k$-algebra on $(Z, *)$ is the quotient space

$$\mathcal{C}_k[Z] := \left(\coprod_{n \geq 0} \mathcal{C}_k(n) \times_{S_n} Z^{\times n} \right) / \sim$$

where $((f_1, \ldots, f_n), (z_1, \ldots, z_{n-1}, *)) \sim ((f_1, \ldots, f_{n-1}), (z_1, \ldots, z_{n-1}))$.

One thinks of this as the space of little k-disks in the unit k-disk, decorated by points in Z. Further, when a little disk is decorated by the basepoint of Z, one may drop it and its label. This is in fact a $\mathcal{C}_k$-algebra: the algebra structure is induced by the operadic composition in $\mathcal{C}_k$ (the decorations get carried along for the ride). In fact, $\mathcal{C}_k[Z]$ is the *free* $\mathcal{C}_k$-algebra on Z in the following sense:

Proposition 8. *There exists a natural bijection*

$$\mathrm{Map}_{\mathcal{C}_k-alg}(\mathcal{C}_k[Z], W) = \mathrm{Map}_{\mathrm{Top}_*}(Z, W).$$

Proof. The map from left to right restricts a $\mathcal{C}_k$-algebra map $\mathcal{C}_k[Z] \to W$ to the subspace

$$Z = \{\mathrm{id}\} \times Z \subseteq \mathcal{C}_k(1) \times Z \subseteq \mathcal{C}_k[Z].$$

In the other direction, for $h : Z \to W$ an arbitrary continuous, pointed map, define $H : \mathcal{C}_k[Z] \to W$ by $H(f, (z_1, \ldots, z_n)) = \theta(f; (h(z_1), \ldots, h(z_n)))$.

□

Define $e : \mathcal{C}_k[Z] \to \Omega^k \Sigma^k Z$ as the free map generated by $Z \to \Omega^k \Sigma^k Z$ which is adjoint to the identity on $\Sigma^k Z$. More prosaically, e carries $((f_1, \ldots, f_n), (z_1, \ldots, z_n))$ to the map $S^k \to \Sigma^k Z$ which wraps the little disk $f_i \subseteq D^k$ around $S^k = \Sigma^k\{*, z_i\} \subseteq \Sigma^k Z$, and carries the rest of $S^k = D^k/\partial$ to the basepoint $*$.

Theorem 9 (Approximation [May72]). *If Z is connected, then e is a weak equivalence.*

We will prove this result in the next lecture. It is needed to prove the recognition theorem. Historically, this result was used to investigate the homology of iterated loop spaces, since $\mathcal{C}_k[Z]$ is more geometrically tractable than $\Omega^k \Sigma^k Z$. One may also reverse the flow of information: from knowledge of the homology of function spaces, we may come to an understanding of the homology of (decorated) configuration spaces.

References

[May72] J. P. May, *The geometry of iterated loop spaces*, Lecture Notes in Mathematics 271, Springer Verlag, Berlin, 1972.