Weeks 2 & 3: Configuration spaces via iterated fibrations

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In these lectures, we'll try to get at the algebraic topology of configuration spaces using some natural fibre sequences. This will inform their homotopy groups (through long exact sequences) and (co)homology (through the Serre spectral sequence).

Functoriality

It is worth noting that the constructions Conf_n and PConf_n are not functors of arbitrary maps $f : X \to Y$. After all, a non-injective map f will carry a set of n distinct points to a set of smaller cardinality. It is, however, a functor of injective continuous maps.

Recall that a homotopy $H : X \times I \to Y$ from f to g is called an ***isotopy*** if H_t is injective for all $t \in I$. It is easy to verify that if f and g are isotopic injective maps, they induce homotopic maps $\mathrm{Conf}_n(X) \to \mathrm{Conf}_n(Y)$.

Let M be a manifold, possibly with boundary. If $\partial M \neq \emptyset$, we will implement the notation

$$\mathrm{Conf}_n(M) := \mathrm{Conf}_n(M \setminus \partial M),$$

and similarly for $\mathrm{PConf}_n(M)$. This is somewhat reprehensible. It is justified by the simplicity of the resulting notation, as well as the following result:

Proposition 1. *Let M be a manifold with boundary, and assume that the boundary is* collared[1]*: there is a neighborhood U of the boundary which is homeomorphic to $\partial M \times (0, 1]$. Then the natural inclusion*

$$\mathrm{Conf}_n(M \setminus \partial M) \subseteq \mathrm{Conf}_n(M)$$

is an equivalence (as is the same map for PConf*).*

[1] This is, for instance, true if M is compact.

Proof. The existence of a collaring of M ensures that there is a continuous injective map $f : M \to M \setminus \partial M$ whose composite with the inclusion $M \setminus \partial M \subseteq M$ is isotopic to the identity. On the collar neighborhood, f is given by shrinking towards 0 in the $(0, 1]$ direction; off the collar, f is the identity. It's easy then to see that the map that f induces configuration spaces is a homotopy inverse to the indicated map.

□

Homeomorphism and diffeomorphism groups

Definition 2. Let M be a manifold; then

$$\mathrm{Homeo}(M) = \{f : M \to M \mid f \text{ is a homeomorphism}\}$$

is a topological group under composition (with the compactly generated compact open topology). If M is smooth, $\mathrm{Diff}(M)$ will denote the subgroup of diffeomorphisms[2].

- If M is orientable, we will write $\mathrm{Homeo}^+(M)$ for the subgroup preserving the orientation.
- If $\partial M \neq \emptyset$, $\mathrm{Homeo}(M, \partial)$ is the subgroup fixing ∂M pointwise.
- If $S \subseteq M$ is a (usually finite) subset of the interior, we will write $\mathrm{Homeo}(M, S)$ for the subgroup fixing S pointwise, and $\mathrm{Homeo}(M)^S$ for the subgroup fixing S as a subset.

Similar notation holds for the corresponding variants on $\mathrm{Diff}(M)$.

Definition 3. The set of components

$$\Gamma(M) := \pi_0 \,\mathrm{Diff}^+(M, \partial)$$

is called the ***(oriented) mapping class group*** of M; this is often also written $\mathrm{Mod}(M)$. When M is a closed, oriented surface Σ_g of genus g, we tend to write $\Gamma_g = \Gamma(\Sigma_g)$. If $\underline{z} \in \Sigma_g$ is a subset of order n, we will write

$$\Gamma_{g,n} := \pi_0 \,\mathrm{Diff}^+(\Sigma_g, \underline{z}) \text{ and } \Gamma_g^n = \pi_0 \,\mathrm{Diff}^+(\Sigma_g)^{\underline{z}}$$

for the ***n-punctured mapping class group*** of Σ_g

There is a natural inclusion $\mathrm{Diff}(M) \subseteq \mathrm{Homeo}(M)$. This is not in general a homotopy equivalence in high dimensions; however, this is true for M closed of dimension less than or equal to 3 [Sma59, Cer68]. Consequently, we have a choice to make when defining the mapping class group in terms of homeomorphisms or diffeomorphisms. As we will essentially only consider this group in dimension 2, this is a distinction that we can safely ignore. This is related to the following pair of results:

[2] There are a variety of alternative topologies that one may equip $\mathrm{Diff}(M)$ with so that, for instance, it becomes a Banach manifold. We will not venture down this road.

Theorem 4 ([Sma59, Hat83]). *If* $n = 2$ *or* 3, $\mathrm{Diff}(D^n, \partial)$ *is contractible.*

We will need the following result, whose proof is a straightforward construction:

Exercise 5. There exists a continuous map $\varphi : D^n \to \mathrm{Diff}(D^n, \partial)$ with the property that $\varphi_z(0) = z$, and $\varphi_z(z) = 0$.

Iterated fibrations

Let $S = \{s_1, \dots, s_m\} \subseteq \{1, \dots, n\}$ be a subset of order m. Define a map

$$p_S : \mathrm{PConf}_n(M) \to \mathrm{PConf}_m(M) \;\text{ by }\; p_S(z_1, \dots, z_n) = (z_{s_1}, \dots, z_{s_m})$$

Proposition 6. *The map* p_S *is a fibre bundle, with fibre over* $\underline{x} \in \mathrm{PConf}_m(M)$ *equal to* $\mathrm{PConf}_{n-m}(M \setminus \underline{x})$.

Proof. For $\underline{x} \in \mathrm{PConf}_m(M)$, let $U_i \ni x_i$ be a small disk such that $U_i \cap U_j = \emptyset$ for $i \neq j$. Then $U = U_1 \times \cdots \times U_m$ is a neighborhood of $\underline{x}$ in $\mathrm{PConf}_m(M)$. For $\underline{x}' \in U$, define $\Phi_{\underline{x}'} \in \mathrm{Homeo}(M, \partial)$ by

$$\Phi_{\underline{x}'}(m) = \begin{cases} m, & m \notin \cup_i U_i \\ \varphi_{x_i'}(m), & m \in U_i. \end{cases}$$

Here we are regarding U_i as the unit disk, with 0 identified with x_i.

Then a homeomorphism $U \times \mathrm{PConf}_{n-m}(M \setminus \underline{x}) \to p_S^{-1}(U)$ is given by sending $(\underline{x}', \underline{z})$ to the appropriate reordering[3] of $(\underline{x}', \Phi_{\underline{x}'}(\underline{z}))$.

□

Corollary 7. *Let* M *be a manifold, and* $\underline{x}$ *an element* $\underline{x} \in \mathrm{PConf}_m(M)$.

1. *There is a long exact sequence*

$$\cdots \longrightarrow \pi_k \,\mathrm{PConf}_{n-m}(M \setminus \underline{x}) \longrightarrow \pi_k \,\mathrm{PConf}_n(M) \xrightarrow{(p_S)_*} \pi_k \,\mathrm{PConf}_m(M) \longrightarrow \cdots$$

2. *If* M *is a connected surface, not equal to* S^2 *or* $\mathbb{R}P^2$, *then* $\pi_k \,\mathrm{PConf}_n(M) = 0$ *for* $k > 1$.

Proof. The first is simply the long exact sequence in homotopy for fibrations. The second follows from the first by induction using the case $m = n - 1$:

$$\cdots \longrightarrow \pi_k \,\mathrm{PConf}_1(M \setminus \underline{x}) \longrightarrow \pi_k \,\mathrm{PConf}_n(M) \xrightarrow{(p_S)_*} \pi_k \,\mathrm{PConf}_{n-1}(M) \longrightarrow \cdots \tag{1}$$

along with the fact that $\mathrm{PConf}_1(M \setminus \underline{x}) = M \setminus \underline{x}$ has vanishing higher homotopy groups whenever M is not S^2 or $\mathbb{R}P^2$.

□

[3] Here, we simply acknowledge that S need not equal $\{1, \dots, m\}$, nor need it be given in that order.

Since the quotient map $\mathrm{PConf}_n(M) \to \mathrm{Conf}_n(M)$ is a Galois covering with deck transformation group S_n, we conclude:

- For $k > 1$, the quotient map $\pi_k \,\mathrm{PConf}_n(M) \cong \pi_k \,\mathrm{Conf}_n(M)$ is an isomorphism.
- There is a (rarely split) short exact sequence

$$1 \longrightarrow \pi_1 \,\mathrm{PConf}_n(M) \longrightarrow \pi_1 \,\mathrm{Conf}_n(M) \longrightarrow S_n \longrightarrow 1 \tag{2}$$

The groups $\pi_1 \,\mathrm{PConf}_n(M)$ and $\pi_1 \,\mathrm{Conf}_n(M)$ are called the *n-strand (pure) braid groups* of M. The homomorphism to S_n may be described by sending a braid to the permutation of its endpoints that it performs.

Example: the configuration space of the plane

Let us compute the first few braid groups of $\mathbb{C}$ (usually just called “the" braid groups).

1. Since $\mathrm{PConf}_1(\mathbb{C}) = \mathrm{Conf}_1(\mathbb{C}) = \mathbb{C}$ is contractible, the first braid group $PB_1 = B_1 = \pi_1(\mathbb{C}) = 1$ is trivial.
2. Using the long exact sequence (1) for $n = 2$ and the previous fact, the inclusion of the fibre (say over 0) is an isomorphism $\pi_1(\mathbb{C} \setminus \{0\}) \cong PB_2$. Of course, $\pi_1(\mathbb{C} \setminus \{0\}) = \pi_1(S^1) = \mathbb{Z}$; this is generated by a loop around 0 of winding number 1. The image of this generator in PB_2 is the braid that winds the first strand around the second, keeping the second fixed.

   The braid group B_2 sits in the exact sequence (2); this is $1 \to \mathbb{Z} \to B_2 \to S_2 \to 1$. In fact, this is the nonsplit extension $1 \to 2\mathbb{Z} \to [B_2 = \mathbb{Z}] \to \mathbb{Z}/2 \to 1$. The generator is represented by the half twist which crosses one strand over the other; it’s square is the generator of PB_2.
3. Generally, using (1) and the fact that $\pi_1(\mathbb{C} \setminus \underline{x}) = F_{n-1}$ is the free group on $n-1$ generators (loops around the punctures $\underline{x}$) when $\underline{x}$ has order $n-1$, there is a short exact sequence

$$1 \longrightarrow F_{n-1} \longrightarrow PB_n \longrightarrow PB_{n-1} \longrightarrow 1 \tag{3}$$

   We will describe this extension in more detail in Proposition 10. In the case $n = 3$, the reader is encouraged to draw a picture of the braids in PB_3 coming from lifts of the generator of PB_2 and the generators of F_2.

Braid groups as mapping class groups

By functoriality, there is an obvious action of $\mathrm{Homeo}(M, \partial)$ (and its various subgroups) on $\mathrm{PConf}_n(M)$ and $\mathrm{Conf}_n(M)$. For any element $\underline{z} \in \mathrm{PConf}_n(M)$, define a continuous map

$$ev_{\underline{z}} : \mathrm{Homeo}(M, \partial) \to \mathrm{PConf}_n(M) \;\text{ by }\; ev_{\underline{z}}(f) = f(\underline{z})$$

Proposition 8. *If M is path connected and of dimension $d > 1$, the map $ev_{\underline{z}}$ is a principal $\mathrm{Homeo}(M, \partial \cup \underline{z})$-bundle, so $ev_{\underline{z}}$ induces a homeomorphism from the coset space*

$$\mathrm{Homeo}(M, \partial)/\,\mathrm{Homeo}(M, \partial \cup \underline{z}) \to \mathrm{PConf}_n(M).$$

Proof. First, we note that $ev_{\underline{z}}$ is surjective. For any configuration $\underline{z}' \in \mathrm{PConf}_n(M)$, a homeomorphism f carrying $\underline{z}$ to $\underline{z}'$ is constructed as follows: find a family of non-intersecting paths γ_i in M from z_i to z_i'. Let U_i be small tubular neighborhoods of these paths, homeomorphic to D^d. Using Exercise 5, there are diffeomorphisms φ_i of U_i carrying z_i to z_i'. Extend these by the identity to a diffeomorphism of M.

Therefore $\mathrm{PConf}_n(M)$ is a principal homogenous $\mathrm{Homeo}(M, \partial)$-space. It is evident that the stabilizer of $\underline{z}$ is $\mathrm{Homeo}(M, \partial \cup \underline{z})$; the orbit-stabilizer theorem yields the indicated isomorphism on the coset space. To lift this to the stronger statement that $ev_{\underline{z}}$ is a fibre bundle, we must construct local trivializations. This is done in much the same way as Proposition 6.

□

This works just as well for $\mathrm{Diff}(M)$ and its subgroups in place of $\mathrm{Homeo}(M)$. We may use the long exact sequence in homotopy for the fibre sequence

$$\mathrm{Diff}^+(M, \partial \cup \underline{z}) \to \mathrm{Diff}^+(M, \partial) \to \mathrm{PConf}_n(M)$$

to relate homotopy groups of these spaces. In particular, we have

Corollary 9. *The space $\mathrm{Diff}^+(D^2, \partial \cup \underline{z})$ has contractible components, and there is a group isomorphism*

$$\Gamma(D^2, \partial \cup \underline{z}) = \pi_0 \,\mathrm{Diff}^+(D^2, \partial \cup \underline{z}) \cong PB_n = \pi_1 \,\mathrm{PConf}_n(D^2).$$

That is, the n^{th} pure braid group is the mapping class group of diffeomorphisms fixing $\underline{z}$ pointwise. Similarly, the full braid group $B_n \cong \Gamma(D^2, \partial)^{\underline{z}}$ is the mapping class group fixing $\underline{z}$ as a set.

Proof. Using Smale's contractibility of $\mathrm{Diff}^+(D^2, \partial)$, the long exact sequence gives an isomorphism

$$\pi_k \,\mathrm{Diff}^+(D^2, \partial \cup \underline{z}) \cong \pi_{k+1} \,\mathrm{PConf}_n(D^2), \;\; k \geq 0.$$

The contractibility of the components follows from the vanishing of these groups when $k > 0$, and the identification of the group of components as the braid group is the content of this isomorphism for $k = 0$.

□

We very much encourage the reader to explore how this theorem fails to be true in the setting where D^2 is replaced with a surface with interesting topology.

Notice that there is an obvious action of $\Gamma(M)$ on $\pi_1(M,*)$, where $* \in \partial M$ by the homotopy (hence isotopy) functoriality of π_1 for maps $M \to M$ fixing the basepoint. We will call this the "natural" action of the mapping class group on $\pi_1(M,*)$. When $M = D^2 \backslash \underline{z}$ is a punctured disk, for instance, this gives an action of $B_n = \Gamma(D^2, \partial)^{\underline{z}}$ on the free group $\pi_1 M \cong F_n$.

Proposition 10. *The short exact sequence (3) is split. In fact PB_n is isomorphic to the semidirect product $PB_{n-1} \ltimes F_{n-1}$ using the natural action of PB_{n-1} on F_{n-1}.*

The proof of this fact employs a number of important ideas in the subject. Recall that (3) is obtained in π_1 from the fibre sequence

$$\mathbb{C} \setminus \underline{x} \xrightarrow{i} \mathrm{PConf}_n(\mathbb{C}) \xrightarrow{p} \mathrm{PConf}_{n-1}(\mathbb{C})$$

where the bundle map forgets the last point in the configuration, and $\underline{x}$ is a configuration of $n-1$ points in $\mathbb{C}$. A splitting to p is given by the ***stabilization map*** $s : \mathrm{PConf}_{n-1}(\mathbb{C}) \to \mathrm{PConf}_n(\mathbb{C})$ which adds a point near infinity. Explicitly, if we identify $\mathbb{C}$ with the interior of the unit disk $D \subseteq \mathbb{C}$, then s is given by

$$s(z_1, \ldots, z_{n-1}) = (\frac{z_1}{2}, \ldots, \frac{z_{n-1}}{2}, \frac{3}{4})$$

(of course, there's nothing special about $3/4$; any element of norm greater than $1/2$ will do). The composite $p \circ s$ scales $\underline{z}$ by a factor of $1/2$, and so is homotopic to the identity. Thus the map s_* induces a splitting of the short exact sequence.

The image of s_* may be identified with those (pure) braids in which the last strand does not move (or more generally, doesn't intertwine with any other strand). For the j^{th} generator f_j of F_{n-1}, the image $i_*(f_j)$ is the braid which keeps the first $n-1$ strands fixed, and winds the last one around the j^{th} strand. The action of PB_{n-1} on F_{n-1} is then by conjugation of these subgroups in PB_n. In fact, this action naturally extends to B_{n-1}. Proposition 10 then follows from:

Exercise 11. Verify that:

1. The action of $\sigma_i \in B_{n-1}$ on $f_j \in F_{n-1}$ by conjugation in PB_n is:

   $$\sigma_i(f_j) = f_j \text{ if } j \neq i, i+1, \quad \sigma_i(f_{i+1}) = f_i, \text{ and } \quad \sigma_i(f_i) = f_i f_{i+1} f_i^{-1}$$

   Your proof should involve pictures of braids.

2. This is precisely the same action as via the identification of B_{n-1} with the mapping class group. **Hint**: the generator σ_i corresponds to the diffeomorphism of D^2 which swaps the i^{th} and $i+1^{\text{st}}$ points in $\underline{z}$ via a Dehn twist along a circle containing these two points.

Cohomology via the Serre spectral sequence

We will compute the cohomology of the ordered configuration spaces of $\mathbb{R}^k$ in this section.

Lemma 12. *The map $f : \mathrm{PConf}_2(\mathbb{R}^k) \to S^{k-1}$ given by $f(x,y) = \frac{x-y}{|x-y|}$ is a homotopy equivalence.*

Proof. Le $\mathbb{R}_{>0}$ denote the multiplicative group of positive real numbers, and $\mathrm{Aff}^+(R^k) = \mathbb{R}_{>0} \ltimes \mathbb{R}^k$ the group of affine transformations of $\mathbb{R}^k$ which scale only by positive numbers. $\mathrm{Aff}^+(\mathbb{R}^k)$ acts on $\mathbb{R}^k$ through homeomorphisms by the formula

$$(a,v) \cdot w = aw + v.$$

We check that for each $(a,v) \neq (1,0)$, there is precisely one fixed point of the action: $aw + v = w$ implies $(a-1)w = -v$. Then either $a = 1$ and $v = 0$ (the identity of $\mathrm{Aff}^+(\mathbb{R}^k)$) or $w = \frac{1}{1-a}v$.

Consequently the induced action of $\mathrm{Aff}^+(\mathbb{R}^k)$ on $\mathrm{PConf}_n(\mathbb{R}^k)$ is without fixed points for $n > 1$. Further, when $n = 2$, f is constant on orbits of $\mathrm{Aff}^+(\mathbb{R}^k)$, and so descends to

$$\overline{f} : \mathrm{PConf}_2(\mathbb{R}^k)/\,\mathrm{Aff}^+(R^k) \to S^{k-1}$$

It's easy to check that $\overline{f}$ is a homeomorphism; an inverse sends z to $(z,0)$. With a little effort, one can show that this makes $\mathrm{PConf}_2(\mathbb{R}^k)$ a principal $\mathrm{Aff}^+(R^k)$-bundle over S^{k-1}. Since the fibre $\mathrm{Aff}^+(R^k)$ is contractible, f is an equivalence.

□

Remark 13. Notice that f is $[S_2 = \mathbb{Z}/2]$-equivariant, where $\mathbb{Z}/2$ acts on S^{k-1} by the antipodal action. This in turn implies that $\mathrm{Conf}_2(\mathbb{R}^k)$ is homotopy equivalent to $\mathbb{R}P^{k-1}$.

Let $i \neq j \in \{1, \ldots, n\}$, and consider the map $p_{ij} : \mathrm{PConf}_n(\mathbb{R}^k) \to \mathrm{PConf}_2(\mathbb{R}^k)$ that projects onto the (i,j) factor. Choose an orientation of S^{k-1}, and let $\omega \in H^{k-1}\,\mathrm{PConf}_2(\mathbb{R}^k) \cong H^{k-1}S^{k-1} = \mathbb{Z}$ be dual to the associated fundamental class. Finally, define

$$\omega_{ij} := p_{ij}^*(\omega) \in H^{k-1}\,\mathrm{PConf}_n(\mathbb{R}^k).$$

Theorem 14 (V. Arnol'd [Arn69] ($k = 2$), F. Cohen [CLM76] ($k \geq 2$)). *There is a ring isomorphism between $H^*\,\mathrm{PConf}_n(\mathbb{R}^k)$ and the quotient of the free graded commutative ring $\mathbb{Z}[\omega_{ij} \,|\, 1 \leq i \neq j \leq n]$ by the relations*

1. $\omega_{ij} = (-1)^k \omega_{ji}$ *for each* $i \neq j$.
2. $\omega_{ij}^2 = 0$ *for each* $i \neq j$.
3. $\omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0$ *for every distinct* i, j, *and* k.

Proof. We first prove this result for $k > 2$. The case $k = 2$ requires a bit more care regarding the simplicity of the local coefficients and the collapsing of the spectral sequence; we address these points at the end of the proof.

That the first two relations hold follows from the corresponding statements about $\omega \in H^{k-1}(S^{k-1})$: its square is certainly zero, and the antipodal map carries it to $(-1)^k\omega$.

The result is clearly true for $n = 1, 2$. Assume inductively that the result is true for $n-1$, and consider the fibre sequence

$$\mathbb{R}^k \setminus \underline{x} \to \mathrm{PConf}_n(\mathbb{R}^k) \to \mathrm{PConf}_{n-1}(\mathbb{R}^k)$$

where $\underline{x}$ has $n-1$ elements. It is the case that the system of local coefficients that the cohomology $H^*(\mathbb{R}^k \setminus \underline{x})$ of the fibre defines on $\mathrm{PConf}_{n-1}(\mathbb{R}^k)$ is simple. This is evidently true when $k > 2$, since $\pi_1 \mathrm{PConf}_{n-1}(\mathbb{R}^k)$ is trivial in that case.

Consequently the Serre spectral sequence for this fibration is of the form

$$E_2^{p,q} = H^p \,\mathrm{PConf}_{n-1}(\mathbb{R}^k) \otimes H^q(\mathbb{R}^k \setminus \underline{x}) \implies H^{p+q} \,\mathrm{PConf}_n(\mathbb{R}^k),$$

Of course $\mathbb{R}^k \setminus \underline{x}$ is homotopy equivalent to the bouquet $\vee_{n-1} S^{k-1}$, so

$$H^q(\mathbb{R}^k \setminus \underline{x}) = \begin{cases} \mathbb{Z}, & q = 0 \\ \mathbb{Z}^{n-1}, & q = k-1 \\ 0, & \text{else.} \end{cases}$$

This is a spectral sequence of algebras. For each r, the r^{th} differential d_r must vanish on $E_r^{*,0}$, since there is nowhere for them to go to. Further, they must vanish on $E_2^{0,*}$: the same reason applies for $* = 0$. For $* = k-1$, inductively we have assumed that $H^* \,\mathrm{PConf}_{n-1}(\mathbb{R}^k)$ is generated by classes in dimension $k-1$. Thus the first possible differential would be

$$d_{k-1} : E_2^{0,k-1} \to E_2^{k-1,1} = H^{k-1} \,\mathrm{PConf}_{n-1}(\mathbb{R}^k) \otimes H^1(\mathbb{R}^k \setminus \underline{x}) = 0.$$

The next possible differential (and all subsequent) must vanish, having target $E_2^{2(k-1),2-k} = 0$ involving negative dimensional cohomology. Since $E_2^{*,*}$ is generated in $E_2^{0,*}$ and $E_2^{*,0}$, all differentials must vanish, and the spectral sequence collapses:

$$H^* \,\mathrm{PConf}_n(\mathbb{R}^k) \cong H^* \,\mathrm{PConf}_{n-1}(\mathbb{R}^k) \otimes H^*(\mathbb{R}^k \setminus \underline{x}).$$

Let's begin to identify the classes represented in this isomorphism. In the first factor, we know by induction that $H^* \,\mathrm{PConf}_{n-1}(\mathbb{R}^k)$ is generated by ω_{ij} for $1 \leq i \neq j \leq n-1$, subject to the indicated relations. In the second factor, $H^{k-1}(\mathbb{R}^k \setminus \underline{x}) \cong \mathbb{Z}^{n-1}$, the i^{th} generator is dual to the homology class corresponding to a small sphere in $\mathbb{R}^k$ around x_i. The image of this sphere in $\mathrm{PConf}_n(\mathbb{R}^k)$ is the family of configurations $\underline{z}$ where $z_j = x_j$ is constant at x_j for $j \neq n$, and z_n moves in a sphere around the fixed point $z_i = x_i$. The image of the dual cohomology class in $H^* \operatorname{PConf}_n(\mathbb{R}^k)$ may therefore be identified with ω_{in}.

Consequently, every element of $H^* \operatorname{PConf}_n(\mathbb{R}^k)$ may be identified as a sum of products of the form

$$\mathbb{Z}\{1, \omega_{1n}, \ldots, \omega_{n-1,n}\} \cdot H^* \operatorname{PConf}_{n-1}(\mathbb{R}^k). \tag{4}$$

Thus $H^* \operatorname{PConf}_n(\mathbb{R}^k)$ is generated by the ω_{ij}. As we have seen that the first two relations hold, it suffices to show the third, and to show that there are no further relations.

To show the third relation, consider the special case $n = 3$; Equation (4) implies

$$H^{2(k-1)} \operatorname{PConf}_3(\mathbb{R}^k) = \mathbb{Z}\{\omega_{13}\omega_{12}, \omega_{23}\omega_{12}\} \cong \mathbb{Z}^2.$$

Now, employing relations (1) and (2) and the graded commutativity of the cup product, every possible product of 2 generators must be (up to sign) one of the following:

$$\omega_{12}\omega_{23}, \quad \omega_{23}\omega_{31}, \quad \text{or} \quad \omega_{31}\omega_{12}.$$

Since the rank of the corresponding cohomology group is 2, there must be a single relation between three classes:

$$a\omega_{12}\omega_{23} + b\omega_{23}\omega_{31} + c\omega_{31}\omega_{12} = 0.$$

For symmetry reasons, however, we must have $a = b = c$: if I apply the permutation $(12) \in S_3$, we get

$$\begin{aligned} 0 &= a\omega_{21}\omega_{13} + b\omega_{13}\omega_{32} + c\omega_{32}\omega_{21} \\ &= a\omega_{12}\omega_{31} + b\omega_{31}\omega_{23} + c\omega_{23}\omega_{12} \\ &= c\omega_{23}\omega_{12} + b\omega_{31}\omega_{23} + a\omega_{12}\omega_{31} \\ &= (-1)^{(k-1)^2}(c\omega_{12}\omega_{23} + b\omega_{23}\omega_{31} + a\omega_{31}\omega_{12}) \end{aligned}$$

So $a = c$. Using another transposition in S_3, we can similarly show that $a = b$. Furthermore, it must be the case that $a = b = c$ is either 1 or -1. This constant can't be 0 (or the rank of the group would be 3), and it can't be of positive norm (or there would be torsion in the group).

Thus relation (3) holds in $H^* \operatorname{PConf}_3(\mathbb{R}^k)$. The general form of relation (3) involves ω_{ij}, ω_{jk}, and ω_{ki}. Thus it may be pulled back from $H^* \operatorname{PConf}_3(\mathbb{R}^k)$ under the projection map $p_S : \operatorname{PConf}_n(\mathbb{R}^k) \to \operatorname{PConf}_3(\mathbb{R}^k)$ where $S = \{i, j, k\}$. So indeed, (3) does hold generally.

To conclude, we must show that there are no further relations than those indicated in the statement of the theorem. We will do so by showing that the 3 indicated relations are sufficient to imply that every element is of the form indicated in (4). As we know that the ω_{ij} generate $H^* \operatorname{PConf}_n(\mathbb{R}^k)$, consider an arbitrary word in the ω_{ij}:

$$w = \omega_{i_1,j_1} \cdots \omega_{i_m,j_m} = \pm\omega_{k_1,n}, \cdots \omega_{k_s,n}\omega_{p_1,q_1} \cdots \omega_{p_{m-s},q_{m-s}}.$$

Here we have collected all of the occurrences of ω_{i_r,j_r} involving $i_r = n$ or $j_r = n$ to the left (using graded commutativity) and rewritten them in the form $\omega_{k_r,n}$ using relation (1). If, for any m, $k_m = k_{m+1}$, then $w = 0$, by relation (2). If this is not the case, using relations (1), (3), and graded commutativity, we have

$$\omega_{k_m,n}\omega_{k_{m+1},n} = \pm\omega_{k_{m+1},n}\omega_{k_{m+1},k_m} \pm \omega_{k_m,n}\omega_{k_{m+1},k_m}.$$

which allows us to reduct the length of of the string of elements of the form $\omega_{k_m,n}$ on the left of the expression for w by 1. Inductively, then, every w is equal to a sum of words beginning with at most one element of the form ω_{kn}; that is, of the form in (4).

For the case $k = 2$, it is not obvious that the system of coefficients defined by $H^*(\mathbb{R}^2 \setminus \underline{x})$ is simple. However, Exercise 11 shows that the action of $\sigma_i \in B_{n-1}$ on $\pi_1(\mathbb{R}^2 \setminus \underline{x})$ is through permutation and conjugation of the generators. In $H_1(\mathbb{R}^2 \setminus \underline{x})$, the B_{n-1}-action therefore factors through S_{n-1}. Thus the restriction to PB_{n-1} is a trivial action in (co)homology.

The argument regarding the collapsing of the spectral sequence (via sparsity) does not hold in this setting, either. However, since the stabilization map $s : \mathrm{PConf}_{n-1}(\mathbb{R}^2) \to \mathrm{PConf}_n(\mathbb{R}^2)$ homotopically splits the projection, the induced map $H^* \mathrm{PConf}_{n-1}(\mathbb{R}^2) \to H^* \mathrm{PConf}_n(\mathbb{R}^2)$ must inject. This is given by the edge map in the spectral sequence; thus no elements in the bottom row can be the target of a differential.

$\square$

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