Midterm for MATH 5345H: Introduction to Topology

October 14, 2013

Due Date: Monday 21 October **in class**.

You may use your book, notes, and old homeworks for this exam. When using results form any of these sources, please cite the result being used. Please explain all of your arguments carefully.

Please do not communicate with other students about the exam. You are free to contact me with questions about the exam at any time.

1. A topological space X is said to be *second countable* if its topology has a basis which is countable. It is a consequence of Problem 8(a) in §13 of Munkres that the real numbers $\mathbb{R}$ are a second countable space, when equipped with the standard topology.

   (a) Show that $\mathbb{R}^n$ is second countable, when equipped with the standard (product) topology.

   (b) Let $Y \subseteq X$ be a subset; give it the subspace topology. If X is second countable, show that Y is, too.

   (c) Assume that if X has a topology given by a subbasis which is countable. Show that X is second countable. Is the converse true?

   (d) Give $\mathbb{R}$ the discrete topology. Is it second countable? Why, or why not?

2. Let X be a totally ordered set.

   (a) Give X the order topology T_{order}; show that (X, T_{order}) is Hausdorff.

   (b) For an element $a \in X$, define the *left ray* to be:

   $$(-\infty, a) := \{x \in X \mid x < a\}$$

   Define B_{left} to consist of left rays, along with the empty set and X.

   $$B_{left} = \{(-\infty, a) \mid a \in X\} \cup \{\emptyset, X\}$$

   Show that B_{left} is a basis for a topology on X, the *left order topology*, T_{left}.

(c) If X has the least upper bound property (Munkres, pg. 27), show that in fact B_{left} is not just a basis, but a topology. Equivalently, $B_{left} = T_{left}$.

(d) Show that T_{order} is strictly finer than T_{left}.

(e) Assume that X has at least two elements, and give it the left order topology. Show that (X, T_{left}) is not Hausdorff.

3. (a) Let K be a set, $L \subseteq K$ a subset, and $f : K \to L$ an injection. Recursively define K_n and L_n (subsets of K) via the rule:

$$K_1 = K, \quad L_1 = L, \quad K_n = f(K_{n-1}), \quad \text{and} \quad L_n = f(L_{n-1}).$$

Show that $L_n \subseteq K_n$, and that $K_n \subseteq L_{n-1}$.

(b) Notice that the previous problem gives a sequence of subsets of K:

$$\cdots \subseteq K_3 \subseteq L_2 \subseteq K_2 \subseteq L_1 \subseteq K_1 = K.$$

This allows us to define a function $g : K \to L$ by

$$g(x) := \begin{cases} f(x) & \text{if } x \in K_n \setminus L_n \text{ for some } n \\ x & \text{otherwise.} \end{cases}$$

Show that g is a bijection.

(c) Now let K and M be any sets, and assume that there are functions $f : K \to M$ and $h : M \to K$ which are injections. Show that there exists a bijection from K to M. **Hint:** The previous part should be helpful.

(d) Let $X = \{0, 1\}$. Recall that we showed that the product X^ω is uncountable (Theorem 7.7 of Munkres). Let Y consist of the set of subsets of X^ω which are countable; that is,

$$Y = \{S \subseteq X^\omega \mid S \text{ is countable}\}$$

Show that there exists a bijection from Y to X^ω. **Hint:** The previous part should be helpful.