Homework #9 for MATH 8301: Manifolds and Topology

November 9, 2017

Due Date: Wednesday 15 November in class.

1. Let X be a path connected topological space, $x_0 \in X$ a basepoint. Let $f : X \to X$ be a continuous map, and assume that $f(x_0) = x_0$. Furthermore, assume that x_0 has a contractible neighborhood $N \subseteq X$. The *mapping torus* of f is the quotient space M_f of $X \times I$ given by

$$M_f := X \times I/(x, 1) \sim (f(x), 0)$$

Let $m_0 = (x_0, 1/2)$ be the basepoint of M_f.

   (a) Show that M_f is path connected.

   (b) Let $U \subseteq M_f$ be the subspace which is the image of $X \times (0, 1)$ under the quotient map. Compute $\pi_1(U, m_0)$ in terms of $\pi_1(X, x_0)$.

   (c) Let $V \subseteq M_f$ be the subspace which is the image of $X \times [0, 1/3) \cup X \times (2/3, 1] \cup N \times I$. Compute $\pi_1(V, m_0)$ in terms of $\pi_1(X, x_0)$ (and another familiar group).

   (d) Compute $\pi_1(U \cap V, m_0)$ in terms of $\pi_1(X, x_0)$.

   (e) Compute $\pi_1(M_f, m_0)$, using the Seifert-van Kampen theorem.

   (f) Let $(G, \cdot)$ be a group, and let $\varphi : G \to G$ be an automorphism of G. We may form the semidirect product $\mathbb{Z} \ltimes G$ as the set of pairs $\mathbb{Z} \times G$, where multiplication is given by

$$(n, g) * (m, h) = (n + m, (\varphi^m(g)) \cdot h)$$

   Show that $\mathbb{Z} \ltimes G$ is isomorphic to the quotient of the free product $\mathbb{Z} * G$ by the relation $tgt^{-1} = \varphi(g)$, where t is the generator of $\mathbb{Z}$.

   (g) Assume now that f is a homotopy equivalence, and let φ be the automorphism of $\pi_1(X, x_0)$ given by $\varphi(\gamma) = f_*(\gamma)$. Show that $\pi_1(M_f, m_0) \cong \mathbb{Z} \ltimes G$.

2. Prove that every continuous map $f : \mathbb{R}P^2 \to S^1$ is nullhomotopic (that is, f is homotopic to a constant function).