Homework #7 for MATH 8301: Manifolds and Topology

October 23, 2017

Due Date: Monday 30 October in class.

1. Let $(X, *)$ be a based topological space, and let $f : S^n \to X$ be a based map. Define a new space Y as the quotient space of $X \sqcup D^{n+1}$ by the equivalence relation $z \sim f(z)$ for $z \in S^n = \partial D^{n+1}$. We describe Y by saying it is gotten from X by *attaching a disk along* f.

   (a) Show that the natural inclusion $X \to Y$ induces an isomorphism $\pi_1(X, *) \cong \pi_1(Y, *)$ if $n > 1$.

   (b) Use the previous to show the following: if Y is a connected d-manifold for $d > 2$ and $y \in Y$, let $B \subseteq Y$ be an open neighborhood of y, homeomorphic to $\mathbb{R}^d$ (and not equal to Y, in case $Y = \mathbb{R}^d$), then the natural inclusion $Y \setminus B \to Y$ induces an isomorphism $\pi_1(Y \setminus B, *) \cong \pi_1(Y, *)$ for any point $* \in Y \setminus B$.

   (c) Let A and B be connected d-manifolds with $d > 2$. Use the previous part to compute the fundamental group $\pi_1(A \# B, *)$ of the connected sum in terms of $\pi_1(A, *)$ and $\pi_1(B, *)$.

2. Recall that $\mathbb{R}P^n$ may be defined as the quotient of S^n by the equivalence relation $x \sim -x$.

   (a) Show that $\mathbb{R}P^n$ is homeomorphic to the quotient of D^n by the relation $x \sim -x$ for every $x \in \partial D^n = S^{n-1}$.

   (b) Use the previous result to show that $\mathbb{R}P^n$ may be constructed by attaching a copy of D^n to $\mathbb{R}P^{n-1}$ via the quotient map $f : S^{n-1} \to \mathbb{R}P^{n-1}$, as in problem 1.

   (c) Show that $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2$ using a polygonal model for $\mathbb{R}P^2$.

   (d) Then, using the results of 1(a) and 2(b), show that $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2$ for all $n > 1$.